Advanced Turbulence Theory Exam

IMP-T

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1 PartI :Derivation

1.

Assume x_n are independent realizations of a random variable with zero mean and it has a Gaussian probability density function. An estimator for the variance is given by:

$$var_N\{x\} = \frac{1}{N}\sum_{n=1}^{N}(x_n - \langle x\rangle)^2 \tag{1.1}$$

Q.

Is this estimator biased? Prove it.

A.

The true ensemble of x is zero,

$$\langle x\rangle = 0 \tag{1.2}$$

Hence the Eq. (1.1) becomes

$$var_N\{x\} = \frac{1}{N}\sum_{n=1}^{N}(x_n)^2 \tag{1.3}$$

The expected value as the true ensemble of this estimator is

$$\begin{aligned}\langle var_N\{x\}\rangle &= \left\langle \frac{1}{N}\sum_{n=1}^{N}(x_n^2)\right\rangle \\ &= \frac{1}{N}\sum_{n=1}^{N}\langle x_2^2\rangle \end{aligned}\tag{1.4}$$

The ensemble of the ensemble is also ensemble,

$$\begin{aligned}\frac{1}{N}\sum_{n=1}^{N}\langle x_2^2\rangle &= \frac{1}{N}N\langle x_n^2\rangle \\ &= \langle x_n^2\rangle \end{aligned}\tag{1.5}$$

where $\langle x_n^2\rangle$ is the true ensemble mean of the variance. Therefore I can conclude this estimator is not biased because the expected value of the estimator is equal to the true ensemble mean.

I got the idea from your note, page 35.

Q.

Derive an expression for the variability of this estimator stating exactly how you used the information provided.

A.

The variability of the estimator ϵ_{F_N} can be expressed as

$$\epsilon_{F_N}^2 = \frac{1}{N}\frac{var\{f\}}{\langle f\rangle^2} \tag{1.6}$$

where F_N is the N samples average of the some function f;

$$F_N = \frac{1}{N}\sum_{n=1}^{N} f_n \tag{1.7}$$

Now I determine the function f as

$$f = (x - \langle x \rangle)^2 \tag{1.8}$$

in order to get F_N as the variance of x;

$$\begin{aligned} F_N &= \frac{1}{N}\sum_{n=1}^{N}(x - \langle x \rangle)^2 \\ F_N &= var_N\{x\} \end{aligned} \tag{1.9}$$

Hence the variability of this estimator is

$$\epsilon^2_{var_N\{x_n\}} = \frac{1}{N}\frac{var\{(x_n - \langle x \rangle)^2}{\langle (x_n - \langle x \rangle)^2 \rangle^2} \tag{1.10}$$

Since $\langle x \rangle = 0$, the equation becomes

$$\begin{aligned} \epsilon^2_{var_N\{x_n\}} &= \frac{1}{N}\frac{var\{x_n^2\}}{\langle x_n^2 \rangle^2} \\ &= \frac{1}{N}\frac{var\{x_n^2\}}{(var\{x\})^2} \end{aligned} \tag{1.11}$$

where $var\{x_n^2\}$ can be modified as

$$\begin{aligned} var\{x_n^2\} &= \langle (x_n^2 - \langle x_n^2 \rangle)^2 \rangle \\ &= \langle (x_n^2 - var\{x\})^2 \rangle \\ &= \langle x_n^4 - 2x_n^2 var\{x\} + (var\{x\})^2 \rangle \\ &= \langle x_n^4 \rangle - 2\langle x_n^2 var\{x\} \rangle + \langle var\{x\}^2 \rangle \end{aligned} \tag{1.12}$$

It can be said that $var\{x\}$ is invariant in an ensemble operation, that means

$$\begin{aligned} \langle x_n^2 var\{x\} \rangle &= var\{x\} \cdot \langle x_n^2 \rangle \\ &= var\{x\} \cdot var\{x\} \\ &= var\{x_n\}^2 \end{aligned} \tag{1.13}$$

Therefore $var\{x_n^2\}$ is

$$var\{x_n^2\} = \langle x_n^4 \rangle - (var\{x_n\})^2 \tag{1.14}$$

Finally

$$\epsilon^2_{var_N\{x\}} = \frac{1}{N}\frac{\langle x_n^4 \rangle - (var\{x_n\})^2}{(var\{x\})^2} \tag{1.15}$$

Now x_n has Gaussian PDF, which means $\langle x_n^4 \rangle = 3(var\{x\})^2$ because the kurtosis of this function is 3; i.e.

$$K = \frac{\langle (x - \langle x \rangle)^4 \rangle}{\langle (x - \langle x \rangle)^2} = 3 \tag{1.16}$$

Eq. (1.15) can be rewritten as

$$\begin{aligned} \epsilon^2_{var_N\{x\}} &= \frac{1}{N}\frac{3(var\{x_n\})^2 - (var\{x_n\})^2}{(var\{x\})^2} \\ &= \frac{2}{N} \end{aligned} \tag{1.17}$$

I got the idea from your note, page 38 and 39.

Q.

If these were digital samples of a continuously varying signal in time, what additional criterion would you need to apply to ensure that samples to be 'effectively' independent? Define what you mean by 'effectively'.

A.

The criterion is the integral scale:

$$T_{int} = \int_0^T \rho(\tau)d\tau \tag{1.18}$$

where $\rho(\tau)$ is the autocorrelation coefficient, T is the length of the record. The variability of the estimator for the mean velocity U_T is

$$\epsilon_{U_T}^2 = \frac{var\{U_T\}}{U^2} \tag{1.19}$$

where U is the true mean value of the velocity. The variance of U_T can be expressed as

$$\begin{aligned} var\{U_T\} &\approx \frac{2var\{u\}}{T}\int_0^T \rho(\tau)d\tau \\ &= \frac{2T_{int}}{T}var\{u\} \end{aligned} \tag{1.20}$$

Hence,

$$\epsilon_{U_T}^2 = \frac{2T_{int}}{T}\frac{var\{u\}}{U^2} \tag{1.21}$$

Comparing Eq.(1.21) to Eq.(1.6), the number of samples N corresponds to T/T_{int}. The 'effectively' number of the samples N_{eff} is

$$N_{eff} = \frac{T}{2T_{int}} \tag{1.22}$$

Now I point that since time is continuous, we can't 'count' time as the number. But if you have certain time as the unity, you can count time at each period. Hence 'effectively' number means that N_{eff} can be handled as the number of independent samples.

I got the idea from your note, page 157, 158 and 159.

2.

In the study of pressure fluctuations the function $1/r$ is always showing up where $r = |\vec{r}|$ is the distance from some given point, say $\vec{x}$.

Q.

Show that this does not have a three-dimensional Fourier transform in the usual sense.

A.

In Cartesian cooedinates the three-dimensional integration of $1/r$ is

$$\int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty}\int_{z=-\infty}^{\infty}\frac{1}{r}dxdydz \tag{1.23}$$

Each direction can be written as the spherical coordinate,

$$x = r\sin\phi\cos\theta \tag{1.24}$$

$$y = r\sin\theta\sin\theta \tag{1.25}$$

$$z = r\cos\phi \tag{1.26}$$

Hence Eq.(1.23) is modified in the spherical coordinate,

$$\begin{aligned}\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{r=0}^{\infty}\frac{1}{r}r^2\sin\phi drd\phi d\theta &= 2\pi\int_{\phi}^{\pi}\int_{r=0}^{\infty}r\sin\phi drd\phi \\ &= 2\pi\left[-\cos\theta\right]_0^{\pi}\int_{r=0}^{\infty}rdr \\ &= 4\pi\int_{r=0}^{\infty}rdr \end{aligned} \tag{1.27}$$

This integral goes to infinity. Therefore, in the usual sense Fourier transform of $1/r$ is impossible.

I got the idea from your note, page 200 saying "*Obviously this integral does not exist. Nor, in fact does its Fourier transform exist (in the ordinary sense)*".

Q.

But see if you can find its Fourier transform in the sense of generalized functions by defining $1/r$ to be the limit of some other integral (or generalized) function. Explain clearly and in detail all steps and assumptions in your derivation.

A.

Using the generalized function,

$$g_L(r) = e^{-\frac{r}{L}} \tag{1.28}$$

The limit of this function at $L \to \infty$ is

$$\lim_{L\to\infty} g_L(r) = 1 \tag{1.29}$$

The Fourier transform of the $f(r) = 1/r$ can be written in the sense of generalized function,

$$\begin{aligned}\mathcal{F}_{gf}\{f(r)\} &= \lim_{L\to\infty}\mathcal{F}\{f(r)g_L(r)\} \\ &= \lim_{L\to\infty}\frac{1}{(2\pi)^3}\iiint_{-\infty}^{\infty}\frac{1}{r}e^{-\frac{1}{r}}e^{-i\vec{r}\cdot\vec{k}}dr_1dr_2dr_3\end{aligned} \tag{1.30}$$

Using the spherical coordinate, the integral part except taking limit becomes

$$\frac{1}{(2\pi)^3}\iiint_{-\infty}^{\infty}\frac{1}{r}e^{-\frac{1}{r}}e^{-i\vec{r}\cdot\vec{k}}dr_1dr_2dr_3 = \frac{1}{(2\pi)^3}\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{r=0}^{\infty}e^{-i\vec{r}\cdot\vec{k}}\frac{1}{r}e^{-\frac{r}{L}}r^2\sin\phi d\theta d\phi dr \tag{1.31}$$

Now I choose the direction of $\vec{k}$ as the same direction as r_3. Dot product of $\vec{k}$ and $\vec{r}$ can be expressed as

$$\vec{k}\cdot\vec{r} = kr\cos\phi \tag{1.32}$$

I got this idea from the discussion with my colleague, Quentin Pellemiel. Substituting Eq.(1.32) to Eq.(1.31) yields

$$\frac{1}{(2\pi)^3}\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{r=0}^{\infty}e^{-ikrcos\phi}e^{-\frac{r}{L}}r\sin\phi d\theta d\phi dr \tag{1.33}$$

Taking the integration along θ, Eq.(1.35) becomes

$$\frac{1}{(2\pi)^3}2\pi\int_{\phi=0}^{\pi}\int_{r=0}^{\infty}r\sin\phi e^{-\frac{r}{L}}e^{-ikr\cos\phi}d\phi dr \tag{1.34}$$

Taking the integration along ϕ,

$$
\begin{aligned}
& \frac{1}{(2\pi)^2}\int_{r=0}^{\infty} re^{-\frac{r}{L}}\left\{\int_{\phi=0}^{\pi}\sin\phi e^{-ikr\cos\phi}d\phi\right\}dr \\
= & \frac{1}{(2\pi)^2}\int_{r=0}^{\infty} e^{-\frac{r}{L}}\left\{\int_{\phi=0}^{\pi}\frac{ikr\sin\phi}{ik}e^{-ikr\cos\phi}d\phi\right\}dr \\
= & \frac{1}{(2\pi)^2}\int_{r=0}^{\infty} e^{-\frac{r}{L}}\left[\frac{1}{ik}e^{-ikr\cos\phi}\right]_0^{\pi}dr \\
= & \frac{1}{(2\pi)^2}\int_{r=0}^{\infty} e^{-\frac{r}{L}}\left[\frac{1}{ik}(e^{ikr}-e^{-ikr})\right]dr \\
= & \frac{1}{(2\pi)^2}\int_{r=0}^{\infty} e^{-\frac{r}{L}}\frac{1}{ik}2i\sin krdr \\
= & \frac{1}{(2\pi)^2}\int_{r=0}^{\infty} e^{-\frac{r}{L}}\frac{2\sin kr}{k}dr
\end{aligned} \tag{1.35}
$$

And then integrating along r,

$$
\begin{aligned}
= & \frac{1}{(2\pi)^2}\int_{r=0}^{\infty} -L\left(e^{-\frac{r}{L}}\right)'\frac{2\sin kr}{k}dr \\
= & \frac{1}{(2\pi)^2}\left[-Le^{-\frac{r}{L}}\frac{2\sin kr}{k}\right]_0^{\infty} - \frac{1}{(2\pi)^2}\int_{r=0}^{\infty} -Le^{-\frac{r}{L}}2\cos krdr \\
= & \frac{1}{(2\pi)^2}(0-0) - \frac{1}{(2\pi)^2}\int_{r=0}^{\infty} L^2\left(e^{-\frac{r}{L}}\right)'2\cos krdr \\
= & -\frac{1}{(2\pi)^2}\left[L^2e^{-\frac{r}{L}}2\cos kr\right]_0^{\infty} + \frac{1}{(2\pi)^2}\int_{r=0}^{\infty} L^2e^{-\frac{r}{L}}2(-k\sin kr)dr \\
= & -\frac{1}{(2\pi)^2}(0-2L^2) - \frac{1}{(2\pi)^2}\int_{r=0}^{\infty} 2L^2k\sin krdr
\end{aligned} \tag{1.36}
$$

From Eq.(1.35) and Eq.(1.36),

$$
\begin{aligned}
\frac{1}{(2\pi)^2}\int_{r=0}^{\infty} e^{-\frac{r}{L}}\frac{2\sin kr}{k}dr + L^2k^2\frac{1}{(2\pi)^2}\int_{r=0}^{\infty} e^{-\frac{r}{L}}\frac{2\sin kr}{k}dr &= \frac{L^2}{2\pi^2} \\
\frac{1+L^2k^2}{(2\pi)^2}\int_{r=0}^{\infty} e^{-\frac{r}{L}}\frac{2\sin kr}{k}dr &= \frac{L^2}{2\pi^2} \\
\frac{1}{(2\pi)^2}\int_{r=0}^{\infty} e^{-\frac{r}{L}}\frac{2\sin kr}{k}dr &= \frac{L^2}{2\pi^2}\frac{1}{1+L^2k^2} \\
&= \frac{1}{2\pi^2\left(\frac{1}{L^2}+k^2\right)}
\end{aligned} \tag{1.37}
$$

Finally, considering limit at $L \to \infty$,

$$
\begin{aligned}
\mathfrak{F}_{gf}\{f(r)\} &= \lim_{L\to\infty}\frac{1}{2\pi^2\left(\frac{1}{L^2}+k^2\right)} \\
&= \frac{1}{2\pi^2k^2}
\end{aligned} \tag{1.38}
$$

3.

The general form for a two-point isotropic function of a scalar field is a bit simpler than for a vector field; e.g. $F_{\theta,\theta}(\vec{k}) = A(k)$only where $A(k)$ is some unknown function of $k = |\vec{k}|$ and $F_{\theta,\theta}(\vec{k})$ is the three-dimensional Fourier transform of the two point scalar correlation function given by $\langle\theta(\vec{x})\theta(\vec{x}+\vec{r})\rangle$.

Q.

Define a one-dimensional spectrum, say $F^{(1)}_{\theta,\theta}(k_1)$ (by integrating over the 2 and 3-directions) and show it is the one-dimensional Fourier transform of $\langle\theta(x, y, z)\theta(x + r, y, z)\rangle$.

A.

The one-dimensional spectrum is defined as

$$F^{(1)}_{\theta,\theta}(k_1) = \iint_{-\infty}^{\infty} F_{\theta,\theta}(\vec{k})dk_2dk_3 \tag{1.39}$$

The three-dimensional spectrum is the Fourier transform of the correlation function given by $B_{\theta,\theta}(\vec{r}) = \langle\theta(\vec{x})\theta(\vec{x}+\vec{r})\rangle$;

$$F_{\theta,\theta}(\vec{k}) = \frac{1}{(2\pi)^3}\iiint_{-\infty}^{\infty} e^{-i\vec{k}\cdot\vec{r}}B_{\theta,\theta}(\vec{r})dr_1dr_2dr_3 \tag{1.40}$$

Substituting Eq.(1.40) to Eq.(1.39) gives

$$F_{\theta,\theta}(\vec{k}) = \iint_{-\infty}^{\infty}\left\{\frac{1}{(2\pi)^3}\iiint_{-\infty}^{\infty} e^{-i\vec{k}\cdot\vec{r}}B_{\theta,\theta}(\vec{r})dr_1dr_2dr_3\right\}dk_2dk_3 \tag{1.41}$$

Now I mention that Fourier transform of 1 is expressed as

$$\frac{1}{2\pi}\int_{\infty}^{\infty} e^{-k_2r_2}dk_2 = \delta(r_2) \tag{1.42}$$

$$\frac{1}{2\pi}\int_{\infty}^{\infty} e^{-k_3r_3}dk_3 = \delta(r_3) \tag{1.43}$$

where δ is Dirac's delta-function. Eq. 1.41 becomes

$$F_{\theta,\theta}(\vec{k}) = \frac{1}{2\pi}\iiint_{-\infty}^{\infty} e^{-ik_1\cdot r_1}B_{\theta,\theta}(r_1, r_2, r_3)\delta(r_2)\delta(r_3)dr_1dr_2dr_3 \tag{1.44}$$

Now I used the following fact:

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0) \tag{1.45}$$

since $\delta(x)$ has a value of 1 at $x = 0$ and otherwise 0. Hence the integrals along r_2 and r_3 are performed, Eq.1.44 becomes

$$F_{\theta,\theta}(\vec{k}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ik_1\cdot r_1}B_{\theta,\theta}(r_1, 0, 0)dr_1 \tag{1.46}$$

Q.E.D

Q.

Now define a three-dimensional spectrum function, say $E_\theta(k)$ by integrating over spherical shells of radius k and show that $A(k) = E(k)/4\pi k^2$.

A.

$E_\theta(k)$ is defined as the integral of $F_{\theta,\theta}(\vec{k})$ over spherical shells of radius $k = |\vec{k}|$; i.e.

$$E(k) = \frac{1}{2}\oint_{k=|\vec{k}|} F_{\theta,\theta}(\vec{k})dS(k) \tag{1.47}$$

where $\oint$ means a surface integral and $dS(k)$ is a spherical area element. In spherical coordinate[1] , Eq.(1.47) becomes

$$E(k) = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} F_{\theta,\theta}(\vec{k})k^2 \sin\phi d\phi d\theta \tag{1.48}$$

Substituting $F_{\theta,\theta}(\vec{k}) = A(k)$ into this equation,

$$\begin{aligned} E(k) &= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} A(k)k^2 \sin\phi d\phi d\theta \\ &= A(k)k^2 \int_{\theta=0}^{2\pi} [-\cos\phi]_0^{\pi} d\theta \\ &= 2A(k)k^2 \int_{\theta=0}^{2\pi} d\theta \\ &= 4\pi k^2 A(k) \end{aligned} \tag{1.49}$$

Finally, A(k) can be expressed as

$$A(k) = \frac{E(k)}{4\pi k^2} \tag{1.50}$$

Q.

Now use these to find the integral and derivative relations between $E_\theta(k)$ and $F^{(1)}_{\theta,\theta}(k_1)$. How do these differ from the ones for an isotropic vector field we derived in class?

A.

One-dimensional spectrum is given by:

$$F^{(1)}_{\theta,\theta}(k_1) = \iint_{-\infty}^{\infty} F_{\theta,\theta}(\vec{k})dk_2dk_3 \tag{1.51}$$

Substituting Eq.(1.50) to Eq.(1.51) gives

$$F^{(1)}_{\theta,\theta}(k_1) = \iint_{-\infty}^{\infty} \frac{E(k)}{4\pi k^2}dk_2dk_3 \tag{1.52}$$

Considering the coordinate,

$$k^2 = k_1^2 + k_2^2 + k_3^3 \tag{1.53}$$

$$k^2 = k_1^2 + \sigma^2 \tag{1.54}$$

where $\sigma^2 = k_2^2 + k_3^2$. They also can be written by:

$$\sigma \sin\theta = k_3 \tag{1.55}$$

$$\sigma \cos\theta = k_2 \tag{1.56}$$

where θ is the angle by k_2 axis and σ^2. The integral becomes

$$\iiint_{k_1,k_2,k_3} dk_1dk_2dk_3 = \iiint_{k_1,\sigma,\theta} \left|\frac{\partial(k_1,k_2,k_3)}{\partial(k_1,\sigma,\theta)}\right| dk_1d\sigma d\theta \tag{1.57}$$

[1]I got the idea from your note, page 205, 209 - 213.

where additional fraction is called Jacobian,

$$\begin{aligned}
\begin{vmatrix} \frac{\partial k_1}{\partial k_1} & \frac{\partial k_1}{\partial \sigma} & \frac{\partial k_1}{\partial \theta} \\ \frac{\partial k_2}{\partial k_1} & \frac{\partial k_2}{\partial \sigma} & \frac{\partial k_2}{\partial \theta} \\ \frac{\partial k_3}{\partial k_1} & \frac{\partial k_3}{\partial \sigma} & \frac{\partial k_3}{\partial \theta} \end{vmatrix} &= \begin{vmatrix} 1 & 0 & 0 \\ 0 & \frac{\partial}{\partial \sigma}\sigma\cos\theta & \frac{\partial}{\partial \theta}\sigma\cos\theta \\ 0 & \frac{\partial}{\partial \sigma}\sigma\sin\theta & \frac{\partial}{\partial \theta}\sigma\sin\theta \end{vmatrix} \\
&= \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sigma\sin\theta \\ 0 & \sin\theta & \sigma\cos\theta \end{vmatrix} \\
&= \sigma\cos^2\theta - (-\sigma\sin^2\theta) \\
&= \sigma\cos^2\theta + \sigma\sin^2\theta \\
&= \sigma \qquad (1.58)
\end{aligned}$$

Hence Eq.(1.52) becomes

$$\begin{aligned}
&\int_{\sigma=0}^{\infty}\int_{\theta=0}^{2\pi} \frac{E(k)}{4\pi k^2}\sigma d\sigma d\theta \\
=\ & 2\pi\int_{\sigma=0}^{\infty} \frac{E(k)}{4\pi k^2}\sigma d\sigma \\
=\ & \int_{\sigma=0}^{\infty} \frac{E(k)}{2k^2}\sigma d\sigma \qquad (1.59)
\end{aligned}$$

For fixed k_1,

$$\begin{aligned}
\sigma^2 &= k^2 - k_1^2 \\
2\sigma d\sigma &= 2kdk \qquad (1.60)
\end{aligned}$$

Subsituting Eq.(1.60) to Eq.(1.59),

$$\begin{aligned}
&\int_{k_1}^{\infty} \frac{E(k)}{2k^2}kdk \\
=\ & \int_{k_1}^{\infty} \frac{E(k)}{2k}dk \qquad (1.61)
\end{aligned}$$

I got this idea from your note, page 213.

Therefore,

$$\begin{aligned}
F_{\theta,\theta}^{(1)}(k_1) &= \int_{k_1}^{\infty} \frac{E(k)}{2k}dk \\
F_{\theta,\theta}^{(1)}(k_1) &= \left[F\left(\frac{E(k)}{2k}\right)\right]_{k_1}^{\infty} \qquad (1.62)
\end{aligned}$$

where $F(\cdot)$ implies the Integral of some function.

$$F_{\theta,\theta}^{(1)}(k_1) = \left[F\left(\frac{E(k)}{2k}\right)\right]_{\infty} - \left[F\left(\frac{E(k)}{2k}\right)\right]_{k_1} \qquad (1.63)$$

where the first term of right hand side goes to 0.

$$\begin{aligned}
F_{\theta,\theta}^{(1)}(k_1) &= -\left[F\left(\frac{E(k)}{2k}\right)\right]_{k_1} \\
\frac{d}{dk_1}F_{\theta,\theta}^{(1)}(k_1) &= -\frac{E(k_1)}{2k_1} \\
E(k) &= -2k\frac{d}{dk}F_{\theta,\theta}^{(1)}(k) \qquad (1.64)
\end{aligned}$$

It has a simpler structure compared to the one of the vector field. The relation of the vector fields is given by

$$E(k) = k^3\frac{d}{dk}\left[\frac{1}{k}\frac{dF_{1,1}^{(1)}}{dk}\right] \qquad (1.65)$$

$$= k^3\left[-\frac{1}{k^2}\frac{dF_{1,1}^{(1)}}{dk} + \frac{1}{k}\frac{d^2F_{1,1}^{(1)}}{dk^2}\right] \qquad (1.66)$$

The first term is completely same as the one I got in Eq. 1.64. We can see that the second order derivative of one-dimensional spectrum doesn't appear in the relation of scalar.

4.

The key to the Pao/Corrsin spectrum was the assumption that the spectral flux, $\varepsilon_k(k) = \alpha_{Kol}^{-1}\varepsilon^{1/3}k^{5/3}E(k)$.

Q.

Plug this into the spectral energy equation for a statistically stationary turbulence and integrate the resulting equation to find the Pao/Corrsin spectrum.

A.

The spectrum energy equation is the following,

$$\frac{\partial}{\partial t}E(k) = T(k) - 2\nu k^2 E(k) \tag{1.67}$$

where $T(k)$ is the energy transfer,

$$T(k) = -\frac{d}{dk}\varepsilon_k \tag{1.68}$$

Now I assume a stationary turbulence which means time dependance can be neglected. Eq.(1.67) becomes

$$0 = -\frac{d}{dk}\varepsilon_k - 2\nu k^2 E(k) \tag{1.69}$$

Plugging $\varepsilon_k(k)$ into this equation,

$$\begin{aligned} \frac{d}{dk}\alpha_{Kol}^{-1}\varepsilon^{1/3}k^{5/3}E(k) &= -2\nu k^2 E(k) \\ \alpha_{Kol}^{-1}\varepsilon^{1/3}\frac{5}{3}k^{2/3}E(k) + \alpha_{Kol}^{-1}\varepsilon^{1/3}k^{5/3}E'(k) &= -2\nu k^2 E(k) \end{aligned} \tag{1.70}$$

The definition of Kolmogorov micro scale is given by

$$\begin{aligned} \eta_{Kol} &= \left(\frac{\nu^3}{\varepsilon}\right)^{\frac{1}{4}} \\ \nu &= \eta_{Kol}^{4/3}\varepsilon^{1/3} \end{aligned} \tag{1.71}$$

Substituting Eq.(1.71) to Eq.(1.70),

$$\begin{aligned} \alpha_{Kol}^{-1}\varepsilon^{1/3}\frac{5}{3}k^{2/3}E(k) + \alpha_{Kol}^{-1}\varepsilon^{1/3}k^{5/3}E'(k) &= -2\eta_{Kol}^{4/3}\varepsilon^{1/3}k^2E(k) \\ \frac{5}{3}\alpha_{Kol}^{-1}k^{2/3}E(k) + \alpha_{Kol}^{-1}k^{5/3}E'(k) &= -2\eta_{Kol}^{4/3}k^2E(k) \end{aligned} \tag{1.72}$$

By dimensional analysis, in an inertial range the energy spectrum[2] is

$$E(k) \sim \varepsilon^{2/3}k^{-5/3} \tag{1.73}$$

I assume the function $E(k)$ as

$$E(k) = C_1\varepsilon^{2/3}k^{-5/3}e^{f(k)} \tag{1.74}$$

[2] It was on your whiteboard

where C_1 is the constant. Plugging Eq.(1.74) to Eq.(1.72),

$$\begin{aligned}
\frac{5}{3}\alpha_{Kol}^{-1}k^{2/3}C_1\varepsilon^{2/3}k^{-5/3}e^{f(k)} + \alpha_{Kol}^{-1}k^{5/3}\{C_1\varepsilon^{2/3}k^{-5/3}e^{f(k)}\}' &= -2\eta_{Kol}^{4/3}k^2C_1\varepsilon^{2/3}k^{-5/3}e^{f(k)} \\
\frac{5}{3}\varepsilon^{2/3}k^{-1}e^{f(k)} + \varepsilon^{2/3}k^{5/3}(-\frac{5}{3}k^{-8/3}e^{f(k)} + k^{-5/3}f'(k)e^{f(k)}) &= -2\eta_{Kol}^{4/3}k^2\alpha_{Kol}\varepsilon^{2/3}k^{-5/3}e^{f(k)} \\
\frac{5}{3}k^{-1} - \frac{5}{3}k^{-1} + f'(k) &= -2\eta_{Kol}^{4/3}k^{1/3}\alpha_{Kol} \\
f(k) &= -2\alpha_{Kol}\eta_{Kol}^{4/3}\frac{k^{4/3}}{4/3} + C_2 \qquad (1.75)
\end{aligned}$$

Finally,

$$\begin{aligned}
E(k) = C_1\varepsilon^{2/3}k^{-5/3}\exp\left(-2\alpha_{Kol}\frac{(\eta_{Kol}k)^{4/3}}{4/3} + C_2\right) \\
E(k) = C\varepsilon^{2/3}k^{-5/3}\exp\left(-2\alpha\frac{(\eta_{Kol}k)^{4/3}}{4/3}\right) \qquad (1.76)
\end{aligned}$$

where $C = C_1 \exp C_2$. In inertial range, where the exponential term's doesn't affect, the spectral flux should be the constant and its value is the same as the dissipation[3]. This fact implies that $E(k) = \alpha_{Kol}\varepsilon^{2/3}k^{-5/3}$. Therefore the constant C is actually $C = \alpha_{Kol}$ and then we obtain

$$E(k) = \alpha_{Kol}\varepsilon^{2/3}k^{-5/3}\exp\left(-2\alpha\frac{(\eta_{Kol}k)^{4/3}}{4/3}\right) \qquad (1.77)$$

Q.

Now plug this into the dissipation integral and integrate it to show that this spectrum integrates to exactly give you the dissipation.

A.

The definition of the dissipation rate is

$$\varepsilon = \int_0^\infty 2\nu k^2 E(k) dk \qquad (1.78)$$

Substituting Eq.(1.76) to Eq.(1.78),

$$\begin{aligned}
\varepsilon &= \int_0^\infty 2\nu k^2\alpha_{Kol}k^{-5/3}\exp\left(-2\alpha_{Kol}\frac{(\eta_{Kol}k)^{4/3}}{4/3}\right)dk \\
&= 2\nu\alpha_{Kol}\int_0^\infty k^{1/3}\exp\left(-2\alpha_{Kol}\frac{(\eta_{Kol}k)^{4/3}}{4/3}\right)dk \\
&= -\nu\int_0^\infty \frac{-2\alpha_{Kol}\eta_{Kol}^{4/3}k^{1/3}}{\eta_{Kol}^{4/3}}\exp\left(-2\alpha_{Kol}\frac{(\eta_{Kol}k)^{4/3}}{4/3}\right)dk \\
&= -\frac{\nu}{\eta_{Kol}^{4/3}}\int_0^\infty \left\{\exp\left(-2\alpha_{Kol}\frac{(\eta_{Kol}k)^{4/3}}{4/3}\right)\right\}' dk \\
&= -\frac{\nu}{\eta_{Kol}^{4/3}}\left[\exp\left(-2\alpha_{Kol}\frac{(\eta_{Kol}k)^{4/3}}{4/3}\right)\right]_0^\infty \\
&= -\frac{\nu}{\eta_{Kol}^{4/3}}[0-1] \\
&= \frac{\nu}{\eta_{Kol}^{4/3}} \qquad (1.79)
\end{aligned}$$

This is exactly the dissipation.

[3] I got the idea from the note of your class.

Q.

What happens when you put it in the energy integral? What is $\langle u_i u_i \rangle$? How do you explain this result?

A.

The definition of the energy integral is

$$\frac{1}{2}\langle u_i u_i \rangle = \int_0^\infty E(k)dk \tag{1.80}$$

Substituting Eq.(1.76) to Eq.(1.80),

$$\frac{1}{2}\langle u_i u_i \rangle = \int_0^\infty \alpha_{Kol}\varepsilon^{2/3}k^{-5/3}\exp\left(-2\alpha_{Kol}\frac{(\eta_{Kol}k)^{4/3}}{4/3}\right)dk \tag{1.81}$$

This value goes to infinity, it means that there is no infinite value as the kinetic energy obtained by this model. The reason why is this model doesn't mention about low wavenumber physics that at low wavenumber energy spectrum goes to zero. Eq.1.86 hasn't a finite number at $k = 0$, that's why the integral of this equation cannot be solved; there is no finite value for the kinetic energy.

5.

Now consider the Lin/Hill model spectrum which is given by:

$$E(k) = \alpha_{Kol}\varepsilon^{2/3}k^{-5/3}[1 + (k\eta_{Kol})^{2/3}]\exp\left\{-2\alpha_{Kol}\left[\frac{(k\eta_{Kol})^{4/3}}{4/3} + \frac{(k\eta_{Kol})^2}{2}\right]\right\} \tag{1.82}$$

Q.

Plug this into the spectral energy equation and work backwards to find out what Lin and Hill assumed for their model of the spectral flux.

A.

I recall the spectral energy equation for the stationary turbulence;

$$\frac{d\varepsilon_k}{dk} = -2\nu k^2 E(k) \tag{1.83}$$

Substituting Eq.(1.82) to Eq.(1.83),

$$\begin{aligned}
\frac{d\varepsilon_k}{dk} &= -2\nu k^2\alpha_{Kol}\varepsilon^{2/3}k^{-5/3}[1 + (k\eta_{Kol})^{2/3}]\exp\left\{-2\alpha_{Kol}\left[\frac{(k\eta_{Kol})^{4/3}}{4/3} + \frac{(k\eta_{Kol})^2}{2}\right]\right\} \\
&= -2\nu\alpha_{Kol}\varepsilon^{2/3}k^{1/3}[1 + (k\eta_{Kol})^{2/3}]\exp\left\{-2\alpha_{Kol}\left[\frac{(k\eta_{Kol})^{4/3}}{4/3} + \frac{(k\eta_{Kol})^2}{2}\right]\right\} \\
&= -2\nu\alpha_{Kol}\varepsilon^{2/3}\eta_{Kol}^{-4/3}[k^{1/3}\eta_{Kol}^{4/3} + k\eta_{Kol}^2]\exp\left\{-2\alpha_{Kol}\left[\frac{(k\eta_{Kol})^{4/3}}{4/3} + \frac{(k\eta_{Kol})^2}{2}\right]\right\}
\end{aligned} \tag{1.84}$$

Integrating along k,

$$\begin{aligned}
\varepsilon_k &= \int \nu\varepsilon^{2/3}\eta_{Kol}^{-4/3}\left(\exp\left\{-2\alpha_{Kol}\left[\frac{(k'\eta_{Kol})^{4/3}}{4/3} + \frac{(k'\eta_{Kol})^2}{2}\right]\right\}\right)' dk' \\
&= \nu\varepsilon^{2/3}\eta_{Kol}^{-4/3}\exp\left\{-2\alpha_{Kol}\left(\frac{(k\eta_{Kol})^{4/3}}{4/3} + \frac{(k\eta_{Kol})^2}{2}\right)\right\} + C
\end{aligned} \tag{1.85}$$

Now I can say $\varepsilon_k(0) = \epsilon$ because in the inertial range the spectrum flux is constant and has the same value as of the dissipation. Strictly speaking, this 0 of $\varepsilon_k(0)$ doesn't mean that the wavenumber is equal to 0 but means that the wavenumber is small enough compared to the Kolmogorov's wavenumber in order to neglect the effect of the exponent. ε_k actually keeps holding the same value from an inertial range to $k = 0$ since this model doesn't consider the shape of the energy spectrum at low wavenumber. Therefore, in this case we can say that $\varepsilon_k(0)$ is the spectrum flux in the inertial range. Using this boundary condition, I get $C = 0$ and

$$\varepsilon_k = \nu\varepsilon^{2/3}\eta_{Kol}^{-4/3} \exp\left\{-2\alpha_{Kol}\left(\frac{(k\eta_{Kol})^{4/3}}{4/3} + \frac{(k\eta_{Kol})^2}{2}\right)\right\} \tag{1.86}$$

Q.

What does this mean? Try to give a physical explanation for the differences from the Pao/Corrsin model above?

A.

The differences of the Pao/Corrsin spectrum are $\{1 + (k\eta_{Kol})^{2/3}\}$ and $\frac{(k\eta_{Kol})^2}{2}$ in the exponential in the energy spectrum. When $k \to 0$, this term goes to 0, hence there is no influence due to these terms. It is important that at high wavenumber the energy spectrum of Lin/Hill model tends to 0 much quicker than the one by Pal/Corrsin model.

Eq.1.86 is expressed using Eq.1.82 as:

$$\varepsilon_k = \alpha_{Kol}^{-1}\varepsilon^{1/3}k^{5/3}E(k)\left[\frac{1}{1 + (k\eta_{Kol})^{2/3}}\right] \tag{1.87}$$

hence $\{1 + (k\eta_{Kol})^{2/3}\}$ appears in the denominator of the energy flux. This function implies that the spectral flux is decreases quicker than the Pao/Corrsin model near the Kolmogorov scale.

2 PartII :Data processing

1.

You have been provided with a set of data acquired by using a hot-wire anemometer in a boundary layer of the LML wind tunnel. Perform the following statistical analyses:

(a) Mean and Variance

Compute the mean and variance using all blocks of data. Subdivide the data into four parts and compute the mean and variance of each part separately. These probably will not be exactly the same, but you will be asked to discuss this below.

A.

The results are shown in the Table.1

Table. 1: Mean and variance

	All blocks	Blocks 1	Blocks 2	Blocks 3	Blocks 4
Mean[m/s]	6.84	6.842	6.850	6.845	6.833
Variance	0.594	0.5976	0.5984	0.5943	0.5840

(b) Spectral Analysis

(i)

Compute the frequency (circular) spectrum (using the FFT and your knowledge of Fortan, etc). Show both linear-linear and log- log plots of your results. Do this three ways: first with just one block of data, then with 10 blocks of data, and finally with all the data. Do you notice any differences if you do not use blocks which have an integer power of 2 (e.g., speed of computation, spurious peaks appearing, etc.)

A.

Fourier transform of the velocity is given by:

$$\hat{u}(f) = \int_{-\infty}^{\infty} u(t)e^{-i2\pi ft}dt \tag{2.1}$$

The spectrum estimator is obtained by:

$$S(f) = \frac{\hat{u}(f)\hat{u}^*(f)}{T} \tag{2.2}$$

First I try to compute the spectrum for each block, and then average them using 10 and 100 blocks. The graphs are shown in Fig.1 through Fig.6. I modified the range of the axis of the frequency in a linear plot in order to see easily. In a linear plot the spectrum estimators have a peak at around 1 and then decrease. The value reaches almost zero at around 100 [Hz]. While in a log-log plot we can see a quite large range of the distribution of the spectrum. The plots using several blocks may seem smoother than the one with only one blocks.

I also try to compute the spectrum with several blocks as a very long record. They are shown in Fig.7 and Fig.8. The spectrum by a long record has a pretty fine scale in the axis of the frequency due to the longer record. The plot looks scattering because of this fineness of the axis of frequency, and because these spectra are not averaged. I note that if these blocks are independent of each other, taking FFT is useless because the end of the record of some block is nothing to do with the beginning of the next block.

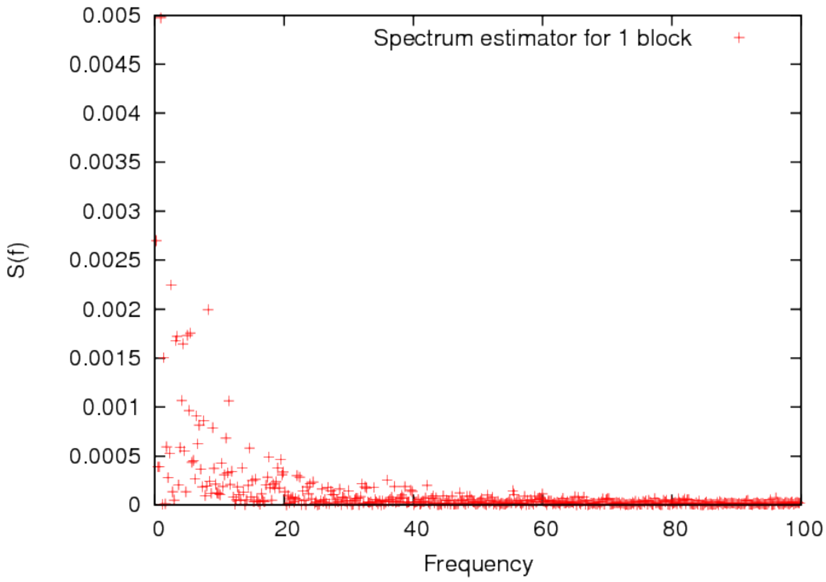

Fig. 1: Spectrum estimator for one block in linear-linear plot

When to take FFT of each block and then average them, each FFT is performed with the elements which has an integer power 2 since each block has 2^{17} number of data. In this case, the number of the blocks doesn't mean anything about the speed of computation etc. by FFT because they have the completely same number of data when to compute FFT. While if I use FFT of blocks as a long record, the number of elements is not always an integer power 2; for instance when you use 10 blocks, the number of data used in FFT is $2^{17} \times 2 \times 5$, which is not an integer power 2. I plot the computation time using several blocks as a long record, which are shown in Fig.9. Obviously the results of 'not' integer power 2 blocks are performed in less efficiency than the one of integer power 2 blocks. The performance of the computation with a big prime number of blocks (or multiples of big prime numbers) are not quite good.

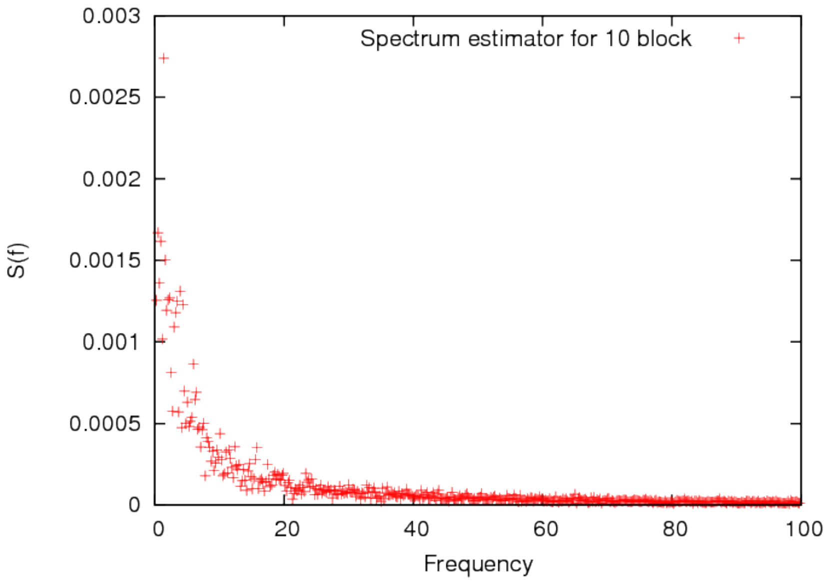

Fig. 2: Spectrum estimator for 10 blocks in linear-linear plot

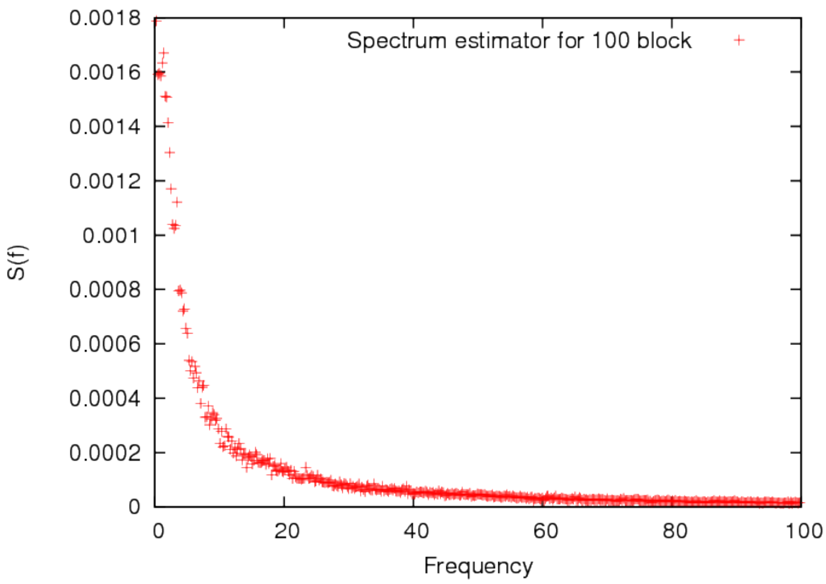

Fig. 3: Spectrum estimator for 100 blocks in linear-linear plot

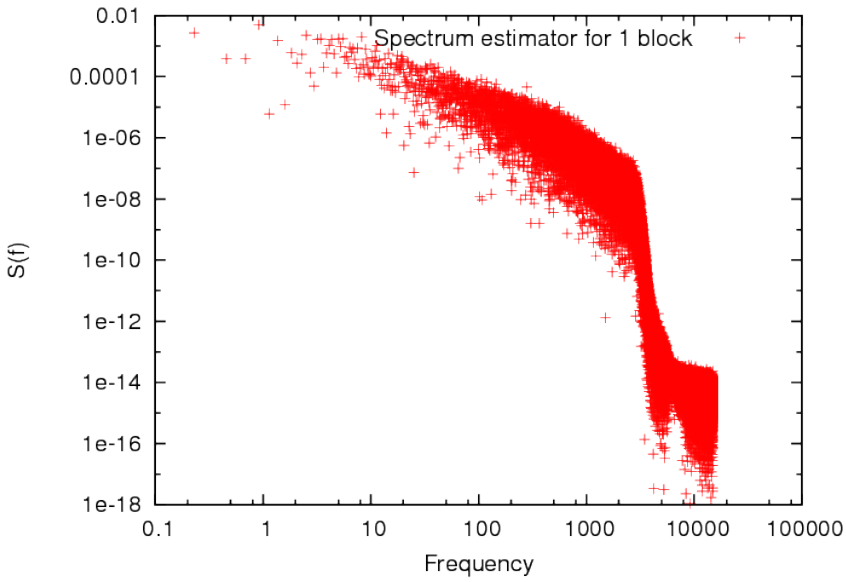

Fig. 4: Spectrum estimator for one block in log-log plot

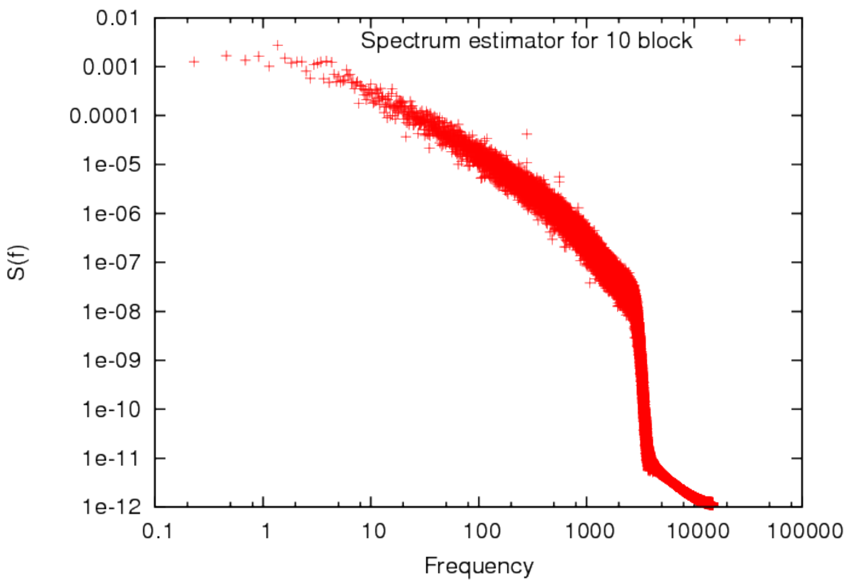

Fig. 5: Spectrum estimator for 10 blocks in log-log plot

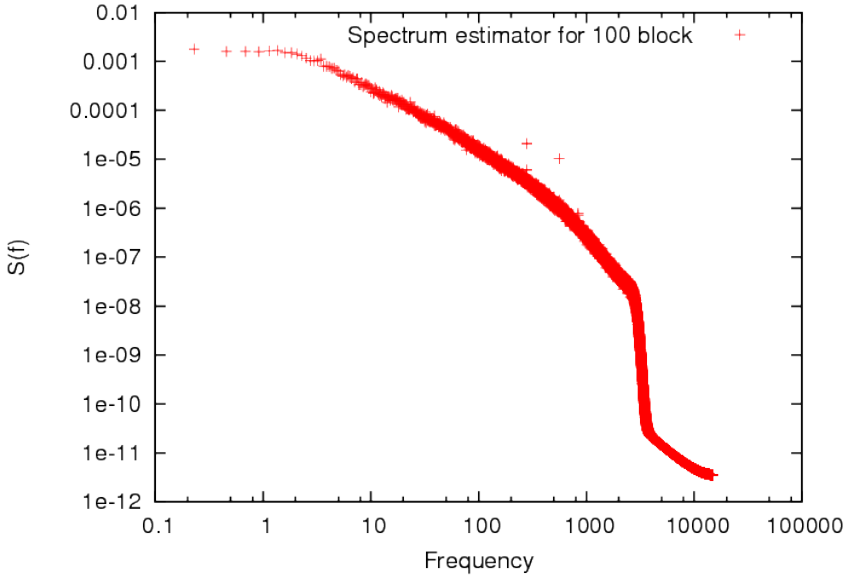

Fig. 6: Spectrum estimator for 100 blocks in log-log plot

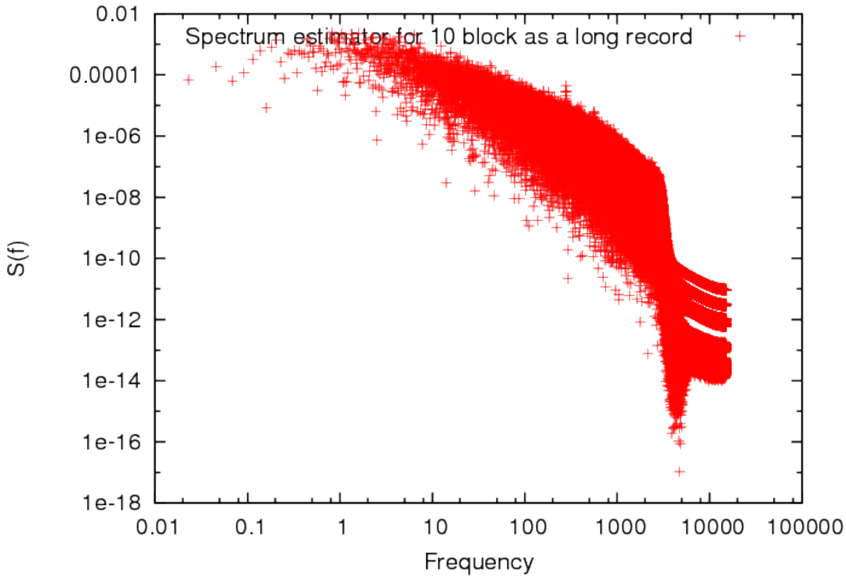

Fig. 7: Spectrum estimator for 10 blocks as a long record in log-log plot

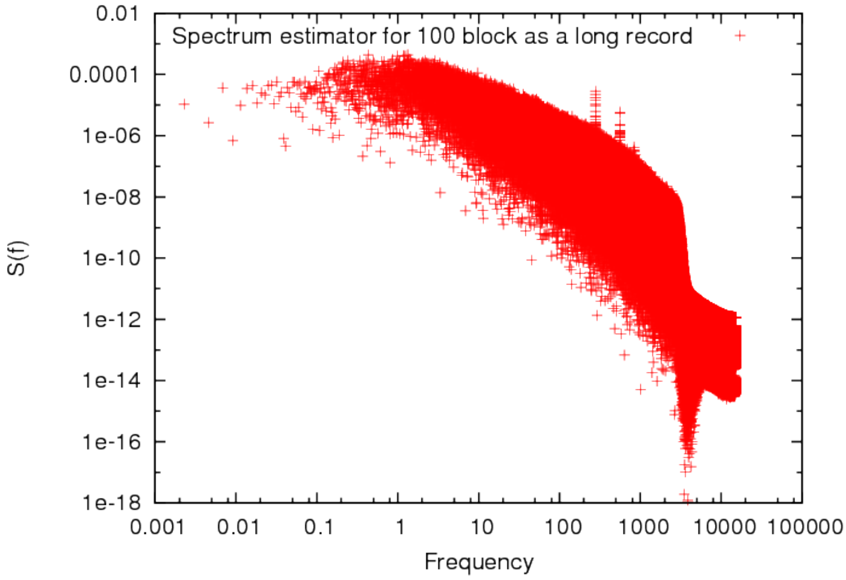

Fig. 8: Spectrum estimator for 100 blocks as a long record in log-log plot

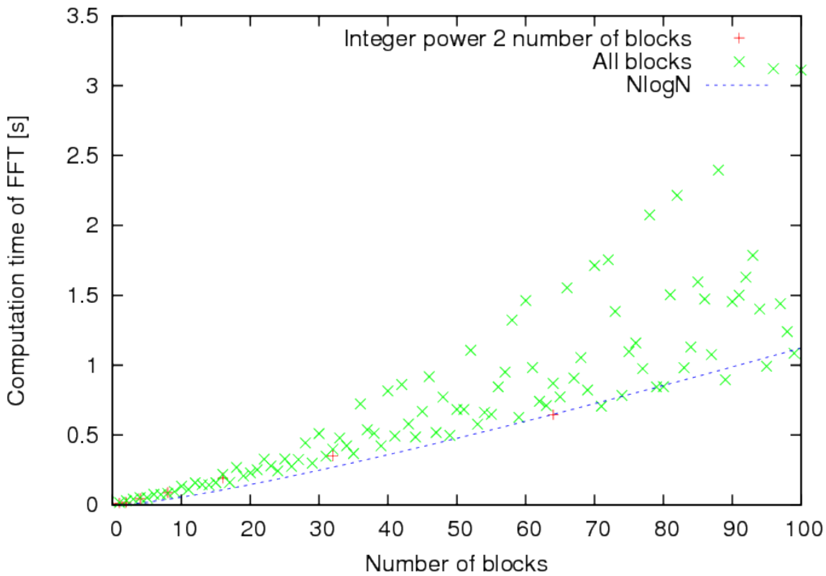

Fig. 9: Computation time of FFT for several blocks

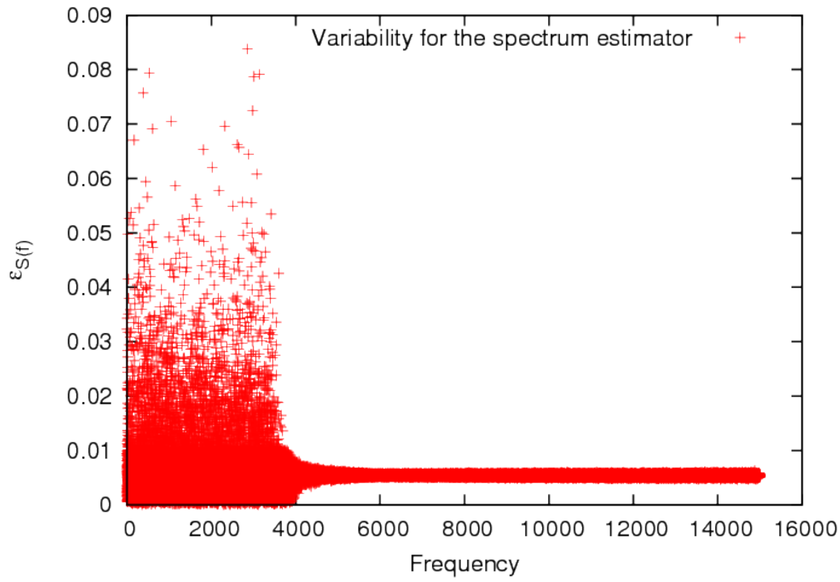

Fig. 10: The variability for the spectrum estimator in linear-linear plot

(ii)

Also compute the variability for your spectral estimator and use it to place error bounds on your spectral estimates at different frequencies. Note how different these look for log-log and lin-lin plots. Why?

A.

The variability of this estimator can be expressed as:

$$\varepsilon^2_{S_N(f)} = \frac{var\{S(f)\}}{\overline{S(f)}^2}\frac{1}{N} \tag{2.3}$$

The variability for the spectrum estimator in linear-linear plot is shown in Fig.10 . In the Fig.10 the variability is distributed around less than 3000 Hz. Above 3000 Hz the variability is quite small since there is no flow structures which cause a fluctuation of the energy (at least they don't have the energy which can appear evidently in the plot) at very high frequencies (means very small scales).

The variability for the spectrum estimator in a log-log plot is shown in Fig.11. In the Fig.11 you can see the variability is more distributed around the high frequencies than around the low frequencies.

The spectrum estimators with error bounds are shown in Fig.12 and Fig.13. In Fig.12 the error bounds looks relatively big at the low frequencies and we can see almost nothing at the high frequencies.

While in Fig.13 the error bounds look quite small almost everywhere. The size of bounds looks same thanks to a log scale in the vertical axis.

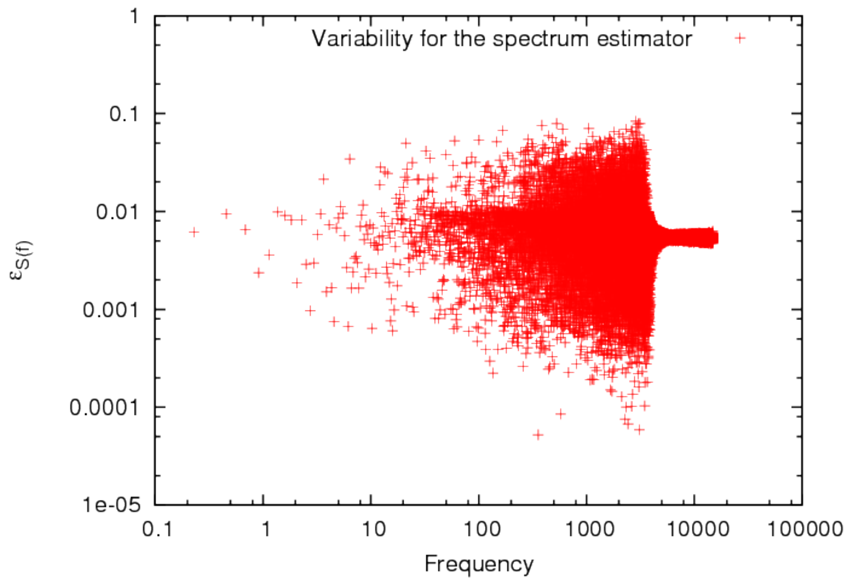

Fig. 11: The variability for the spectrum estimator in log-log plot

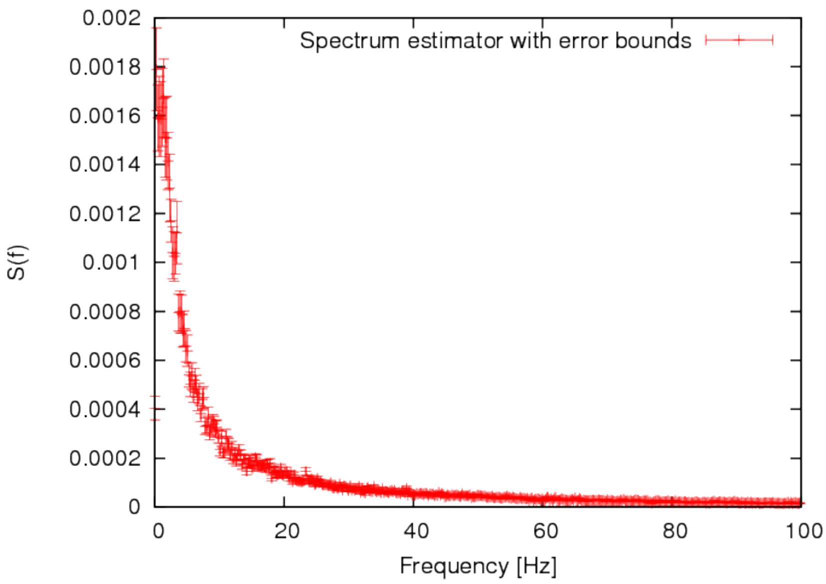

Fig. 12: Spectrum estimator with error bounds in lin-lin plot

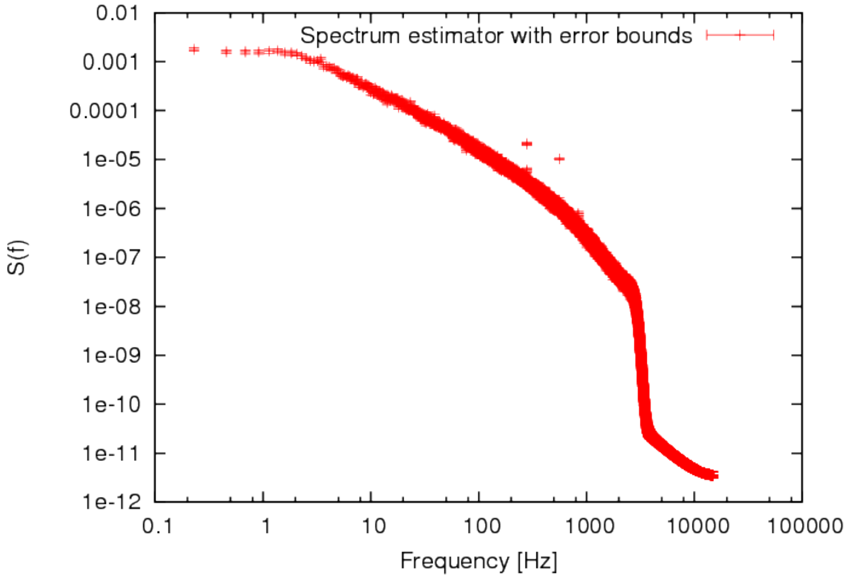

Fig. 13: Spectrum estimator with error bounds in log-log plot

(iii)

Compare the integral under the spectrum with the variance computed in above. Try to explain or eliminate any discrepancies (hint: Look for obvious evidence of noise. And don't forget you should be working with a whole line spectrum so only half of the energy is at positive frequencies. Where are the negative ones hiding?). If different which should you believe to be the most reliable.

A.

The autocorrelation is the inverse Fourier transform of the spectrum,

$$B(\tau) = \int_{-\infty}^{\infty} e^{i2\pi f\tau} S(f) df \tag{2.4}$$

The autocorrelation can be expressed as:

$$B(\tau) = \langle u(t)u(t+\tau)\rangle \tag{2.5}$$

The definition of the variance is given by:

$$var\{u\} = \langle u(t)u(t)\rangle \tag{2.6}$$

This is the case of $\tau = 0$. Eq.(2.4) becomes

$$\langle u(t)u(t)\rangle = \int_{-\infty}^{\infty} S(f) df \tag{2.7}$$

Since the spectrum is an even function;

$$\begin{aligned} var\{u(t)\} &= 2\int_{0}^{\infty} S(f) df \\ &= 0.031 \end{aligned}$$

I eliminate singular peaks at around 300 Hz and 500 Hz. The variance computed above is 0.594. There is a large difference between them.

I try to think what the difference is and which one we should believe. In the direct method, we have $2^{17} \times 100$ number of samples to compute the variable. The variability of the variance is quite small due to this big number of samples. While in the spectral method, the integral is determined almost only by the value at the low frequencies; nevertheless, we have a pretty big error at the low frequencies as we saw in Fig.12. The reason why is that the Fourier transfer in order to obtain the value at the low frequencies has only 100 samples to compute the average; each frequency has 100 samples as well. Actually the maximum variability of the spectrum estimator is the order of around 10^{-1}. And we should use a some method, such as the rectangle method that I used, to integrate under the spectrum; the error might come from the integration as well. For these reasons I conclude that the variance from the direct method is more reliable than the one by the spectral method.

(iv)

Now use the your spectrum and variance to estimate the integral scale (using the intercept as $f \to 0$). How accurate would you guess this estimate to be (based on the accuracy of your variance and spectrum and extrapolation)?

A.

The spectrum $S(f)$ is the Fourier transform of the autocorrelation $B(\tau)$, which is given by:

$$S(f) = \int_{\infty}^{\infty} e^{-i2\pi f\tau} B(\tau) d\tau \tag{2.8}$$

When $f \to 0$, Eq.(2.8) becomes

$$S(0) = \int_{\infty}^{\infty} B(\tau) d\tau \tag{2.9}$$

Since the autocorrelation function is an even function;

$$\int_{\infty}^{\infty} B(\tau) d\tau = 2 \int_{0}^{\infty} B(\tau) d\tau \tag{2.10}$$

The definition of the autocorrelation coefficient $\rho(\tau)$ is

$$\rho(\tau) \equiv \frac{B(\tau)}{B(0)} \tag{2.11}$$

$B(0)$ is known as variance; i.e.

$$B(0) = var\{u\} \tag{2.12}$$

Substituting Eq.(2.11) and Eq.(2.12) to Eq.(2.10) gives

$$S(0) = 2var\{u\} \int_{0}^{\infty} \rho(\tau) d\tau \tag{2.13}$$

The integral scale is given by:

$$T_{int} = \int_{0}^{\infty} \rho(\tau) d\tau \tag{2.14}$$

Therefore the integral scale can be determined as

$$\begin{aligned} T_{Int} &= \frac{S(0)}{2var\{u\}} \\ &= 1.67 \times 10^{-3} \text{ s} \end{aligned} \tag{2.15}$$

Using a logarithmic method,

$$\frac{\delta T_{Int}}{T_{Int}} = \left| \frac{\delta S(0)}{S(0)} \right| + \left| \frac{\delta varu}{varu} \right| \tag{2.16}$$

The relative error of the variance of the variable is the order of 10^{-4} (we will see it in later problem). The relative error of S(0) is affected by the error of the spectrum and the extrapolation. The error of the spectrum is the order of 10^{-2}. I used the first order extrapolation to obtain S(0).

I found the following relation [4] which is given by:

$$e(x) = \frac{f^{(n+1)}(\xi)}{(n+1)\,!}\prod_{i=0}^{n}(x - x_i) \tag{2.17}$$

where ξ is the point given by the mean value theorem. When the first order approximation n=1, the equation becomes

$$e(x) = \frac{f''(\xi)}{2\,!}(x - x_1)(x - x_0) \tag{2.18}$$

where x is the extrapolated point, x_i is the nodes that we already have. Hence the error is proportional to the second order derivative at the point according to the mean value theorem and a distance. The distance is

$$(x_1 - 0)(x_0 - 0) = 0.22 \times 0.45 \sim 0.1 \tag{2.19}$$

We don't know the order of the derivative. I assume that the error mostly depends on the distance.

Since the error of the spectrum is the order of 10^{-2}, the relative error of the spectrum is about 0.1. According to Eq.2.16,

$$\frac{\delta T_{Int}}{T_{Int}} = 0.1 + 0.01 \sim 0.1 \tag{2.20}$$

Therefore the order of the relative error of the integral scale is 0.1.

(c) Correlation analysis

(i)

Compute the two-point correlation directly from the data by averaging the products of two velocities at different times.

A.

The two-point correlation is shown in Fig.14

(ii)

Estimate the statistical error for several time lags and show them on your plot.

A.

The variability of the estimator for the autocorrelation is given by:

$$\epsilon^2_{B_N(\tau)} = \frac{1}{N}\frac{var\{B(\tau)\}}{\overline{B(\tau)}^2} \tag{2.21}$$

where N is the number of blocks, $B(\tau)$ is autocorrelation and the bar means the average by all the blocks. The variance of the autocorrelation is expressed as:

$$var\{B(\tau)\} = \frac{1}{N}\sum_{i=1}^{N}\left(B(\tau)_i - \overline{B(\tau)}\right)^2 \tag{2.22}$$

The plot is shown in Fig.15. Since the errors are quite small, you can see almost nothing.

[4]http : //en.wikipedia.org/wiki/Polynomial_interpolation

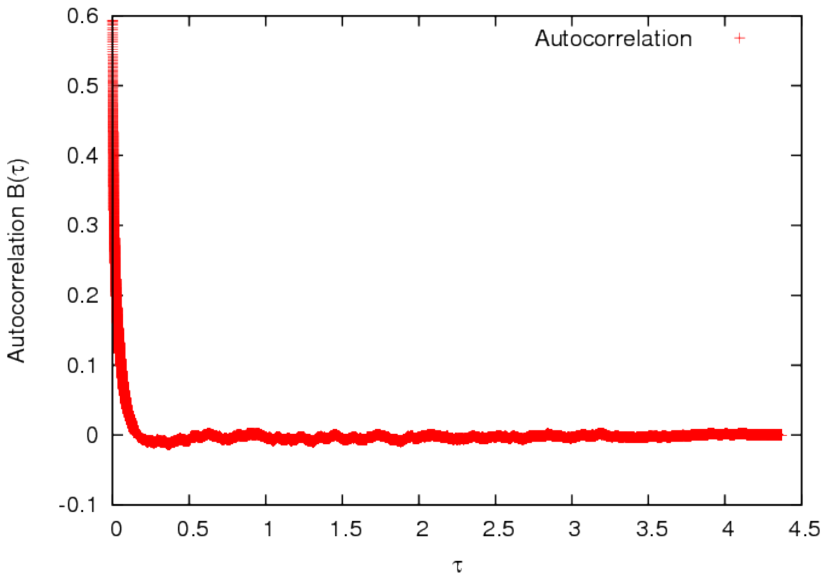

Fig. 14: Autocorrelation

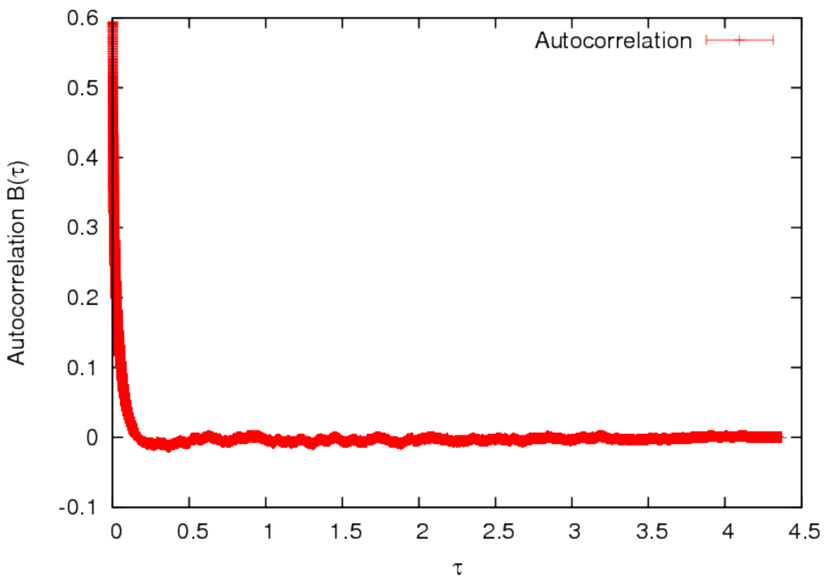

Fig. 15: Autocorrelation coefficient with error bounds

(iii)

Now estimate the integral time scale by integrating under the autocorrelation function, and compare the integral scale obtained in this manner to the one you estimated from the spectra above. Which do you believe to be the most accurate? Why?

A.

By the direct method the integral scale is given by:

$$\begin{aligned} T_{int} &= \int_0^\infty \rho(\tau) d\tau \\ &= 6.52 \times 10^{-3} \text{ s} \end{aligned} \tag{2.23}$$

The accuracy of the integral scale depends on the autocorrelation coefficient and the method when to integrate. The statistical error of the autocorrelation is quite small. For that reason I believe the value by the direct method.

(iv)

Now compute the autocorrelation by taking the inverse Fourier transform of the spectrum you computed above (Remember where the negative frequencies are). Plot them together and explain any differences. If different, which should you believe. Explain why you believe this.

A.

The spectrum is the Fourier transform of the autocorrelation, which is given by:

$$S(f) = \int_{\infty}^{\infty} e^{-i2\pi f\tau} B(\tau) d\tau \tag{2.24}$$

In the Fig.6 only positive frequencies can be seen. Negative frequencies are hiding as the line symmetry. The correlation obtained by the Fourier transform is shown in Fig.16 as the autocorrelation coefficient. In Fig.16 there is two peaks on both side of the plot duo to Fourier transform; later half of plots correspond to the negative time gap $-\tau$, but we want to know the only first half of the plots (Fig.17). Fig.18 shows autocorrelation coefficients by both the spectrum and the direct method. The difference is quite small[5]. But the reliability of $B(0)$ of the direct method is higher than the one of the spectrum. But we already discussed about it above. I believe the autocorrelation by the direct method.

(v)

Now fit the oscullating parabola to your autocorrelation and estimate from it the Taylor microscale.

A.

I used the least-squares method to fit the parabola. The results are shown in Fig.19. Teylor micro scales are obtained as

$$\lambda_{direct} = 1.34 \times 10^{-3} [\text{s}]$$
$$\lambda_{spectrum} = 1.19 \times 10^{-3} [\text{s}]$$

[5]In your class you said the difference comes from the factor of $(1 - |\tau|/T)$ but I found out the main error actually came from my mistake about the way to use FFT. I computed again with the correct way of FFT and obtain the good autocorrelation shown in Fig.18

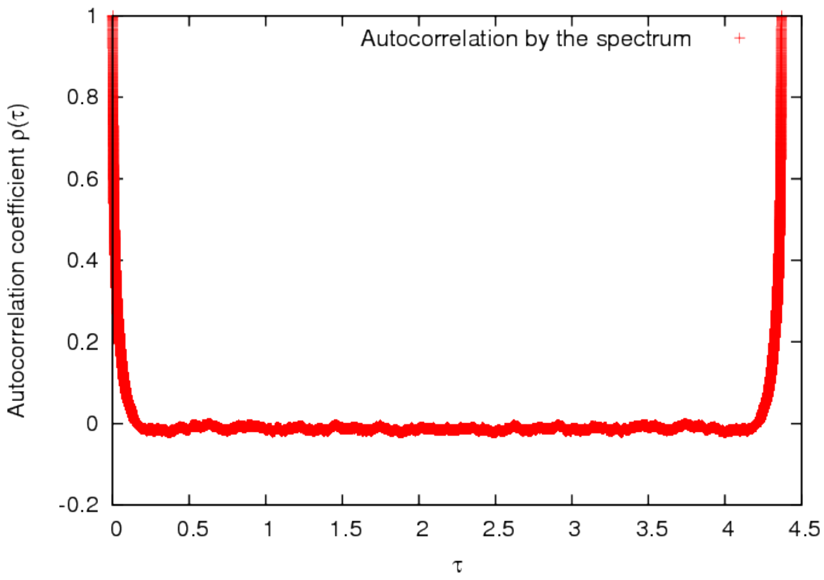

Fig. 16: Autocorrelation coefficient from the spectrum

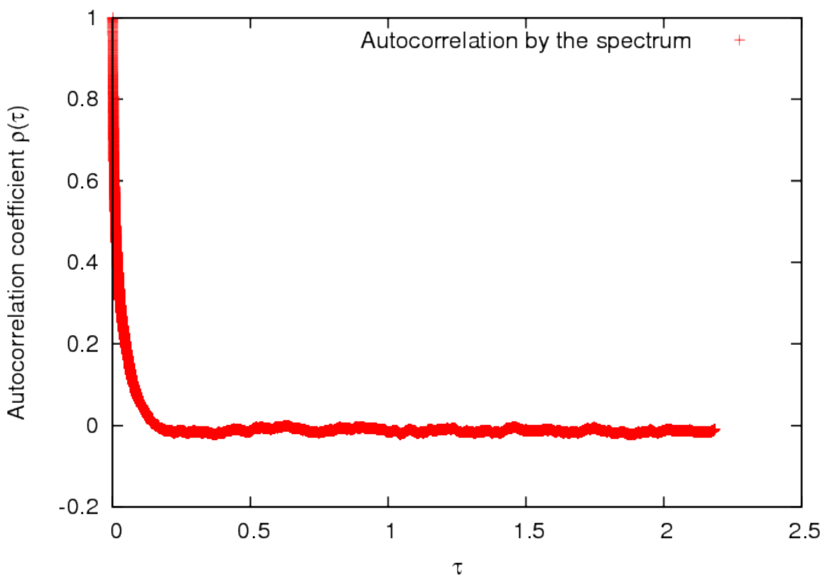

Fig. 17: Half part of the autocorrelation coefficient from the spectrum

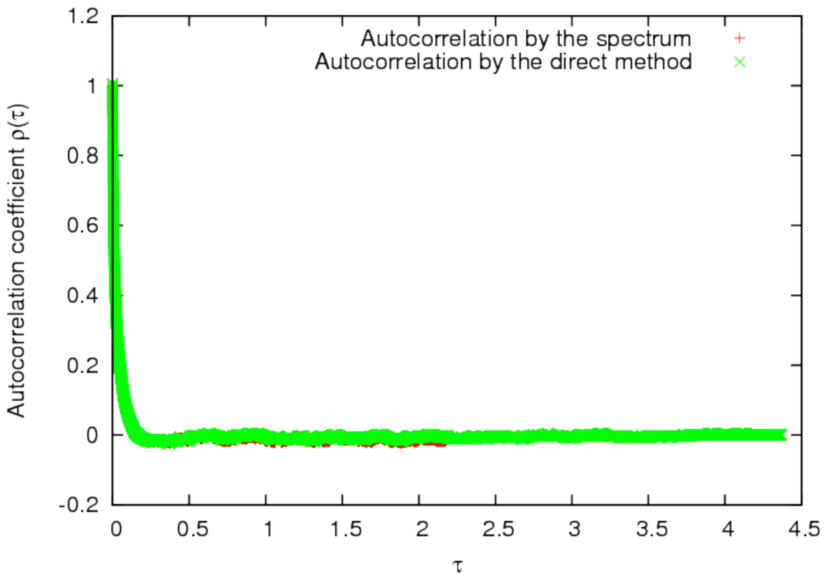

Fig. 18: Autocorrelation coefficient from the spectrum and the direct method

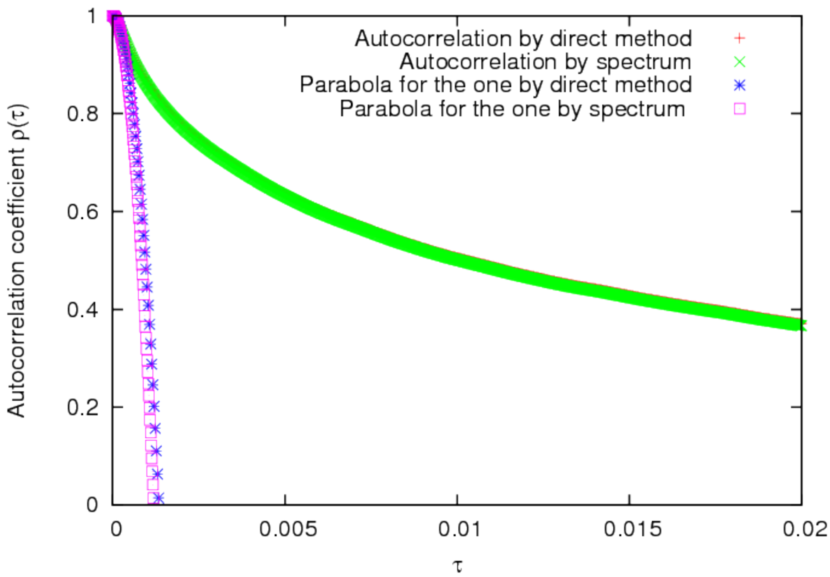

Fig. 19: Fitting parabola to the autocorrelations

(d) Variability of Mean and Variance

(i)

You have all the information to compute the variability for each of your estimators (it will be different for the entire record and the shorter versions), and use it to place error bounds on your results. The means and variances are all likely to be different.

A.

The variabilities for the estimator of mean and variance are given by

$$\varepsilon^2_{X_N} = \frac{1}{N}\frac{var\{x\}}{\langle x\rangle^2} \tag{2.25}$$

$$\varepsilon^2_{var_N\{x\}} = \frac{1}{N}\frac{\langle x^4\rangle - (var\{x\})^2}{(var\{x\}))^2} \tag{2.26}$$

respectively. The results are collected in the Table.2

Table. 2: Variability of mean and variance

	All blocks	Blocks 1	Blocks 2	Blocks 3	Blocks 4
Variability for Mean	3.11×10^{-5}	6.24×10^{-5}	6.24×10^{-5}	6.22×10^{-5}	6.18×10^{-5}
Variability for Variance	3.55×10^{-4}	7.12×10^{-4}	7.03×10^{-4}	7.12×10^{-4}	7.14×10^{-4}

And Table.3 shows the results which are bounded by these errors.

Table. 3: Mean and variance with error

	Mean[m/s]	Variance
All blocks	$6.84\pm2.13\times10^{-4}$	$0.594\pm2.11\times10^{-4}$
Blocks 1	$6.842\pm4.27\times10^{-4}$	$0.5976\pm4.25\times10^{-4}$
Blocks 2	$6.850\pm4.27\times10^{-4}$	$0.5984\pm4.21\times10^{-4}$
Blocks 3	$6.845\pm4.26\times10^{-4}$	$0.5943\pm4.23\times10^{-4}$
Blocks 4	$6.833\pm4.22\times10^{-4}$	$0.5840\pm4.17\times10^{-4}$

(ii)

Use your variability estimates to decide whether these numbers are really different (meaning the mean and variances changed during the run), or whether you can account for the difference by just the statistical errors arising from the finite length of your record. (Hint: In the variability estimate you have to decide whether to use sqrt (1/N) for the number of independent samples, or sqrt [(2 Int/T)] where T is the block length and Int is the integral scale. Explain the difference, and then decide which to use, explaining your choice.

A.

In stationary random process, the record length is divided by twice of the integral scale and treated as the independent number of samples. According to this, the effective number of samples is expressed as

$$N_{eff} = \frac{T}{2T_{Int}} \tag{2.27}$$

where $T = 4.37$ [s] and $T_{Int} = 1.67\times10^{-3}$ [s]. The effective number of sample is $N_{eff} = 1308$. I also compute the variability of the estimator shown in Table.4.

Table. 4: Variability of mean and variance by N_{eff}

	All blocks	Blocks 1	Blocks 2	Blocks 3	Blocks 4
Variability for Mean	3.11×10^{-4}	6.25×10^{-4}	6.25×10^{-4}	6.23×10^{-4}	6.18×10^{-4}
Variability for Variance	3.56×10^{-3}	7.13×10^{-3}	7.04×10^{-3}	7.13×10^{-3}	7.15×10^{-3}

Table. 5: Mean and variance with error by N_{eff}

	Mean[m/s]	Variance
All blocks	$6.84\pm2.13\times10^{-3}$	$0.594\pm2.11\times10^{-3}$
Blocks 1	$6.842\pm4.28\times10^{-3}$	$0.5976\pm4.26\times10^{-3}$
Blocks 2	$6.850\pm4.28\times10^{-3}$	$0.5984\pm4.21\times10^{-3}$
Blocks 3	$6.845\pm4.26\times10^{-3}$	$0.5943\pm4.24\times10^{-3}$
Blocks 4	$6.833\pm4.22\times10^{-3}$	$0.5840\pm4.18\times10^{-3}$

And Table.5 collects the results bounded by these errors. I should believe the variables from N_{eff} because the difference of the mean and variance during the run is the order of magnitude of about 10^{-3}. It corresponds to the variabilities I obtain by N_{eff}.

(e) Taylor's Hypothesis

(i)

Use Taylor's hypothesis to convert your spectra to wavenumber spectra.

A.

Using Talor's hypothesis the wavenumber can be expressed as

$$k = \frac{2\pi f}{U} \tag{2.28}$$

where U is the mean velocity. The spectrum is given by:

$$S(f) = \int_{-\infty}^{\infty} e^{-i2\pi f\tau} B(\tau) d\tau \tag{2.29}$$

First, I recall the scaling property of the Fourier transfer[6],

$$\mathcal{F}\{f(at)\} = \frac{1}{|a|} F\left(\frac{f}{a}\right) \tag{2.30}$$

Applying this for the Eq.(2.29),

$$S\left(\frac{f}{a}\right) = |a| \int_{-\infty}^{\infty} e^{-i2\pi f\tau} B(a\tau) d\tau \tag{2.31}$$

Let $a = \frac{U}{2\pi}$, we obtain the follows:

$$\begin{aligned} S\left(\frac{2\pi f}{U}\right) &= \frac{U}{2\pi} \int_{-\infty}^{\infty} e^{-i2\pi f\tau} B\left(\frac{U}{2\pi}\tau\right) d\tau \\ S(k) &= \frac{U}{2\pi} \int_{-\infty}^{\infty} e^{-i2\pi f\tau} \left\langle u(t) u\left(t + \frac{U}{2\pi}\tau\right)\right\rangle d\tau \\ &= \frac{U}{2\pi} S(f) \end{aligned} \tag{2.32}$$

where I assume $B(\tau) = B(\frac{U}{2\pi}\tau)$ because τ (or $\frac{U}{2\pi}\tau$) might have all the possible time lags. The spectrum is shown in Fig.20.

[6]http : //en.wikipedia.org/wiki/Fourier_transform

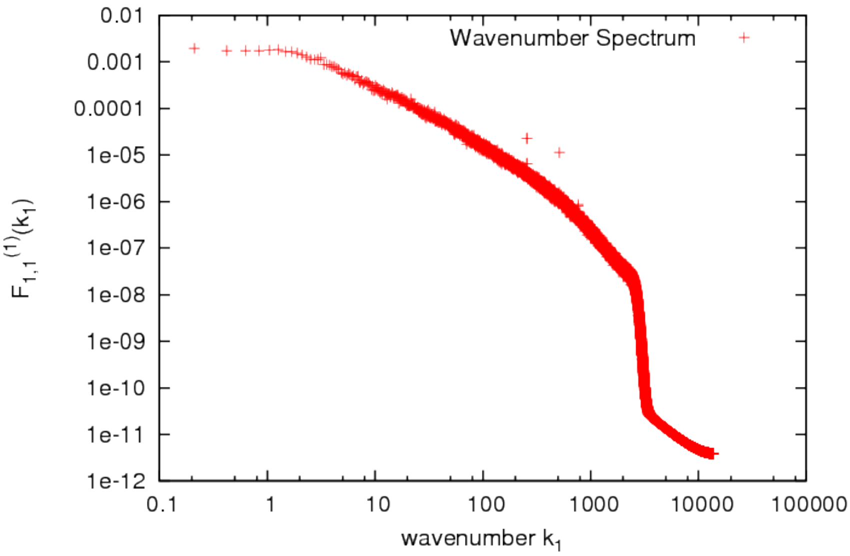

Fig. 20: Wavenumber spectrum by Taylor's hypothesis

(ii)

Identify a $k^{-5/3}$ region in your spectrum above, and use it together with Kolmogorov ' s ideas to obtain an estimate for the rate of dissipation of turbulence energy per unit mass, ϵ. (Hint: plot $k_1^{5/3}F_{1,1}(k_1)$ versus k_1.) State clearly any assumptions you had to make about the data in order for the theory you applied to be valid.

A.

The graph of $k_1^{5/3}F_{1,1}^{(1)}(k_1)$ versus k_1 is shown in Fig.21. The $k^{-5/3}$ region is where the plots are parallel to the horizontal axis, around $k = 250$ to 500. First I assume that the flow is isotropic to introduce the relation between $F_{1,1}^{(1)}(k_1)$ and $E(k)$ given by:

$$F_{1,1}^{(1)}(k_1) = \frac{1}{2}\int_{k_1}^{\infty}\frac{E(k)}{k^3}[k^2 - k_1^2]dk \tag{2.33}$$

According to Kolmogorov's idea, in the inertial range the energy spectrum is given by:

$$E(k) = \alpha_K\epsilon^{2/3}k^{-5/3} \tag{2.34}$$

Substituting Eq.2.34 to Eq.2.33,

$$\begin{aligned} F_{1,1}^{(1)}(k_1) &= \frac{1}{2}\int_{k_1}^{\infty}\frac{\alpha_K\epsilon^{2/3}k^{-5/3}}{k^3}[k^2 - k_1^2]dk \\ &= \frac{\alpha_K\epsilon^{2/3}}{2}\int_{k_1}^{\infty}\left(k^{-8/3} - k^{-14/3}k_1^2\right)dk \\ &= \frac{\alpha_K\epsilon^{2/3}}{2}\left[-\frac{3}{5}k^{5/3} + \frac{3}{11}k^{-11/3}k_1^2\right]_{k_1}^{\infty} \\ &= -\frac{\alpha_K\epsilon^{2/3}}{2}\left(\frac{-33+15}{55}k_1^{-5/3}\right) \\ &= \frac{9}{55}\alpha_K\epsilon^{2/3}k_1^{-5/3} \end{aligned}$$

When I plot $k_1^{5/3}F_{1,1}^{(1)}(k_1)$, the cooeficient C at the inertial range might be the same as $\frac{9}{55}\alpha_K\epsilon^{2/3}$, hence the dissipation rate is obtained by:

$$\epsilon = \left(\frac{55}{9\alpha_K}C\right)^{\frac{3}{2}} \tag{2.35}$$

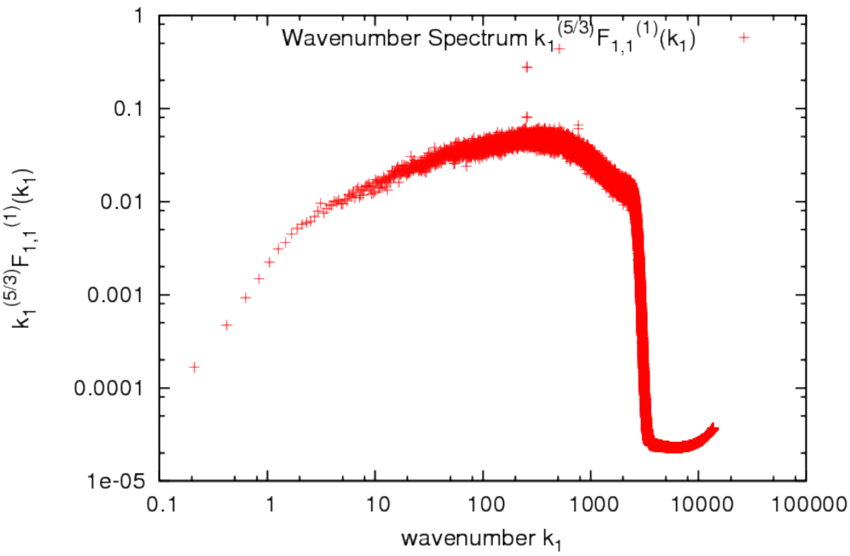

Fig. 21: $k_1^{5/3} F_{1,1}^{(1)}(k_1)$

Now I assume $\alpha_K = 1.5$ and I get 0.047 as C from Fig.21. Finally I obtain $\epsilon = 0.0838$.

(iii)

Now use the value of ϵ you estimated to estimate the Kolmogov microscale. The hot-wire was about 2mm long. Assuming it can resolve scale to about twice its length, decide whether you could have determined the dissipation from the spectrum by integrating $k^2 F_{1,1}^{(1)}$. To what value of $k\eta_K$ do you think your measurements are valid?

(A.)

I assume that Reynolds number is high enough so that we can consider a local isotropy. Kolmogorov's idea is that the small scale motions of turbulence are determined by the Kolmogorov's scale η_{Kol}, the energy dissipation rate ϵ and kinematic viscosity ν. The Kolmogorov scale is given by:

$$\begin{aligned} \eta_K &= \left(\frac{\nu^3}{\epsilon}\right)^{\frac{1}{4}} \\ &= 4.48 \times 10^{-4} \text{ [m]} \end{aligned}$$

The scale which can be resolved by the hot wire is $\eta = 4 \times 10^{-3}$ [m]. The value of $k_1\eta_K$ we want is gives by

$$\frac{k_1\eta_K}{k_1\eta} = 0.11 \tag{2.36}$$

Therefore the measurement is valid up to $k_1\eta_K = 0.11$.

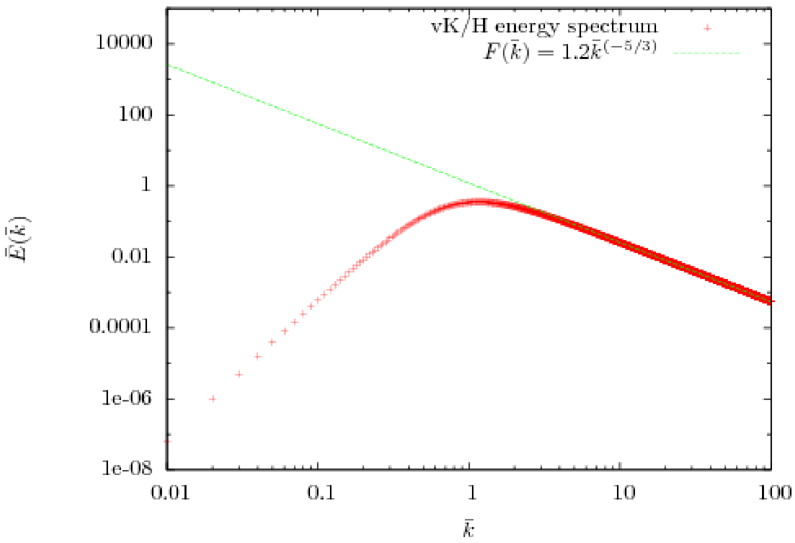

Fig. 22: vK/H energy spectrum $\bar{E}(\bar{k})$

3 PartIII : Turbulence

2.

In class I suggested that you really could only expect there to be a real equilibrium range with constant spectral flux if the turbulence at these wavenumbers could be assumed to be statistically stationary and L_ϵ/η_{Kol} was greater than about 10^3 or $R_L = uL_\epsilon/\nu > 10^4$. Assuming $L_\epsilon = u^3/\epsilon$, use the von Karman/Howarth and Lin/Hill spectra to evaluate this idea.

Assume the vK/H three-dimensional energy spectrum model to be given in energy variables by:

$$\bar{E}(\bar{k}) = \frac{E}{u^2 L} = \frac{C_p \bar{k}^4}{\{1 + (\bar{k}/\bar{k}_e)^2\}^{17/6}} \tag{3.1}$$

where L is the physical integral scale,$\bar{k} = kL$, $Cp = 6.25$ and $\bar{k}_e = 0.747$.

(a)

Make a log-log plot of this and identify with a straight line what a $k^{-5/3}$ range would look like.

A.

The plot of $\bar{E}(\bar{k})$ is shown in Fig.22. The straight line fits to the energy spectrum and shows the place where the inertial range is located.

(b)

At what value of k can you assume to have reached the inertial subrange to within 1% ? 3% ? 10%. Make a plot $k^{5/3}E$ and show these points on it.

A.

The graph of $\bar{k}^{5/3}\bar{E}(\bar{k})$ is shown in Fig.23. The plot reaches at 1.2, which is the value of the limit of this function. The inertial range is indicated as a plateau. The table.6 shows the $\bar{k}$ with 1%, 3% 10% margin to the inertial range. They are also shown in Fig.23 as dot lines.

Table. 6: Wavenumber which reaches inertial subrange within some margin

	$\bar{k}$	$\bar{k}^{5/3}\bar{E}(\bar{k})$
1%	12.6	1.1848
3%	7.2	1.1610
10%	3.9	1.0771

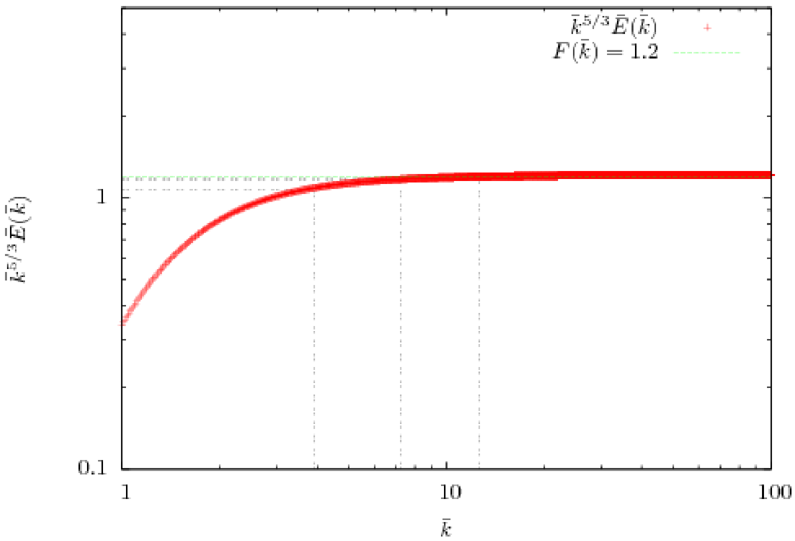

Fig. 23: $\bar{k}^{5/3}\bar{E}(\bar{k})$

(c)

Show that this integrates (for the values given) to 100 % of the energy.

A.

In isotropic turbulence the energy is expressed as

$$\frac{3}{2}u^2 = \int_0^\infty E(k)dk \tag{3.2}$$

where $E(k)$ can be expressed as

$$E(k) = u^2 L\bar{E}(\bar{k}) \tag{3.3}$$

where $\bar{k} = kL$. Using $d\bar{k} = Ldk$, Eq.3.2 becomes

$$\begin{aligned} \frac{3}{2}u^2 &= u^2\int_0^\infty \bar{E}(\bar{k})d\bar{k} \\ \frac{3}{2} &= \int_0^\infty \bar{E}(\bar{k})d\bar{k} \end{aligned} \tag{3.4}$$

The integral of this energy spectrum is expressed as

$$\int_0^\infty \bar{E}(\bar{k})d\bar{k} = \int_0^\infty \frac{C_p\bar{k}^4}{\left\{1+\left(\frac{\bar{k}}{\bar{k}_e}\right)^2\right\}^{\frac{17}{6}}}d\bar{k} \tag{3.5}$$

When I change the variable from $\bar{k}' = \bar{k}/\bar{k_e}$ and $d\bar{k}' = \frac{d\bar{k}}{\bar{k_e}}$, Eq.3.5 becomes

$$\int_0^\infty \frac{C_p \bar{k_e}^5 \bar{k}'^4}{\left\{1+\bar{k}'^2\right\}^{\frac{17}{6}}} d\bar{k}' \tag{3.6}$$

Changing the variable as $x = \frac{1}{1+\bar{k}'^2}$ and $2kdk = -\frac{1}{x^2}dx$,

$$\begin{aligned} & C_p \bar{k_e}^5 \int_1^0 \frac{1}{2} x^{17/6} \left(\frac{1-x}{x}\right)^{3/2} \left(\frac{-1}{x^2}\right) dx \\ = & \frac{C_p \bar{k_e}^5}{2} \int_0^1 (1-x)^{3/2} x^{-2/3} dx \\ = & \frac{C_p \bar{k_e}^5}{2} \int_0^1 (1-x)^{\frac{5}{2}-1} x^{\frac{1}{3}-1} dx \end{aligned} \tag{3.7}$$

where the integral can be written as a beta function [7],

$$\int_0^1 (1-x)^{\frac{5}{2}-1} x^{\frac{1}{3}-1} dx = B\left(\frac{1}{3}, \frac{5}{2}\right) \tag{3.8}$$

Beta function can be expressed as a product of the Gamma function[8],

$$B\left(\frac{1}{3}, \frac{5}{2}\right) = \frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{1}{3}+\frac{5}{2}\right)} \tag{3.9}$$

$$= \frac{\Gamma\left(\frac{1}{3}\right)\frac{3}{4}\Gamma\left(\frac{1}{2}\right)}{\frac{55}{36}\Gamma\left(\frac{5}{6}\right)} \tag{3.10}$$

Now considering the following property of the Gamma function,

$$\Gamma(2z)\,\pi^{-1/2} 2^{2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)} \tag{3.11}$$

Let $z = \frac{1}{3}$,

$$\Gamma\left(\frac{2}{3}\right) = \pi^{-1/2} 2^{2/3-1} \Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{5}{6}\right) \tag{3.12}$$

Hence $\Gamma\left(\frac{5}{6}\right)$ can be written by:

$$\Gamma\left(\frac{5}{6}\right) = 2^{\frac{1}{3}} \sqrt{\pi}\,\Gamma\left(\frac{2}{3}\right) \tag{3.13}$$

And also using the following relation;

$$\Gamma(1-z) = \Gamma(z) = \frac{\pi}{\sin(\pi z)} \tag{3.14}$$

Let $z = \frac{1}{3}$,

$$\Gamma\left(\frac{2}{3}\right) = \frac{2\pi}{\sqrt{3}\,\Gamma\left(\frac{1}{3}\right)} \tag{3.15}$$

Eq.3.10 becomes

$$\frac{27}{55} \frac{\sqrt{3}\,\Gamma\left(\frac{1}{3}\right)^3}{2^{\frac{4}{3}}\pi} \tag{3.16}$$

[7]H. Wang and W. K. George, 2002

[8]http : //en.wikipedia.org/wiki/Beta_function

where $\Gamma\left(\frac{1}{3}\right) = 2.6789...$[9] Therefore,

$$\begin{aligned} \frac{27}{55}\frac{\sqrt{3}\Gamma\left(\frac{1}{3}\right)^3}{2^{\frac{4}{3}}\pi} &= \frac{27}{55}\frac{\sqrt{3}2.6789^3}{2^{\frac{4}{3}}\pi} \\ &= 2.0649 \end{aligned}$$

Substituting this value and the constants to Eq.3.7,

$$\begin{aligned} &\frac{C_p\bar{k_e}^5}{2}2.0649 \\ = \ &1.5009 \end{aligned} \tag{3.17}$$

Q.E.D.

(d)

What is the relation between the L and the k_ϵ above.

A.

Using the relation of the integral scale,

$$\frac{2}{\pi} = C_p \int_0^\infty \frac{\bar{k}^3}{\left\{1+\left(\frac{\bar{k}}{\bar{k_e}}\right)^2\right\}^{17/6}} d\bar{k} \tag{3.18}$$

I change the variable as

$$\begin{aligned} \bar{k}' &= \frac{\bar{k}}{\bar{k_e}} \\ d\bar{k}' &= \frac{d\bar{k}}{\bar{k_e}} \end{aligned}$$

Using this,

$$\frac{2}{\pi} = C_p\bar{k_e}^4 \int_0^\infty \frac{\bar{k}'^3}{\left\{1+\bar{k}'^2\right\}^{17/6}} d\bar{k} \tag{3.19}$$

Again, I choose x as

$$\begin{aligned} x &= \frac{1}{1+\bar{k}'^2} \\ 2\bar{k}'d\bar{k}' &= -\frac{1}{x^2}dx \end{aligned}$$

Eq.3.19 becomes

$$\begin{aligned} \frac{2}{\pi} &= C_p\bar{k_e}^4 \int_1^0 \frac{1}{2}x^{17/6}\left(\frac{1-x}{x}\right)\left(-\frac{1}{x^2}\right)dx \\ &= \frac{C_p\bar{k_e}^4}{2}\int_0^1 (1-x)^{2-1}x^{5/6-1}dx \\ &= \frac{C_p\bar{k_e}^4}{2}B\left(2,\frac{5}{6}\right) \end{aligned} \tag{3.20}$$

[9]http://en.wikipedia.org/wiki/Particular_values_of_the_gamma_function

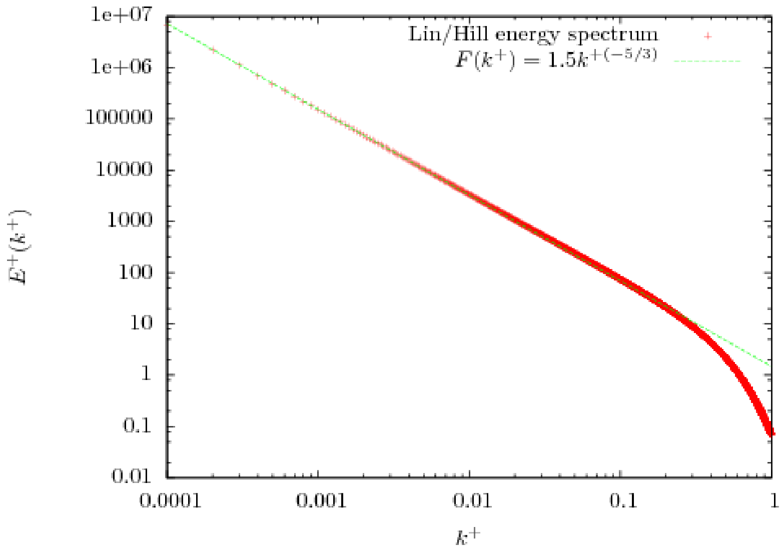

Fig. 24: Lin/Hill energy spectrum

From Eq.(3.7) and Eq.(3.20),

$$\frac{6}{4}\pi = \bar{k_e}\frac{B\left(\frac{1}{3},\frac{5}{2}\right)}{B\left(2,\frac{5}{6}\right)} \tag{3.21}$$

After few lines of a simple calculus, we obtain:

$$\sqrt{\pi} = \bar{k_e}\frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{5}{6}\right)} \tag{3.22}$$

Now we know $\bar{k_e} = k_e \times L$, finally we obtain the relation between k_e and L;

$$k_e = \frac{\sqrt{\pi}\Gamma\left(\frac{5}{6}\right)}{L\Gamma\left(\frac{1}{3}\right)} \tag{3.23}$$

(3)

Assume the Lin/Hill spectrum model to be given in Kolmogorov variables by:

$$E^+(k^+) = \frac{E}{\nu^{5/4}\epsilon^{1/4}} = \alpha_K k^{+^{-5/3}}[1 + k^{+^{2/3}}]\exp\{-\alpha_K[(3/2)k^{+^{-4/3}} + k^{+^{-2}}]\} \tag{3.24}$$

where $k^+ = k\eta_{Kol}$ and $\alpha_K = 1.5$

(a)

Make a log-log plot of this, again identifying the $k^{5/3}$ range. Up to what wavenumbers is it reasonable to assume a $k^{5/3}$ range to within 1%, 3% 10% ?

A.

The graph of $E^+(k^+)$ and $k^{+^{5/3}}E^+(k^+)$ are shown in Fig.24 and Fig.25. The reasonable range is estimated and shown in Table.7.

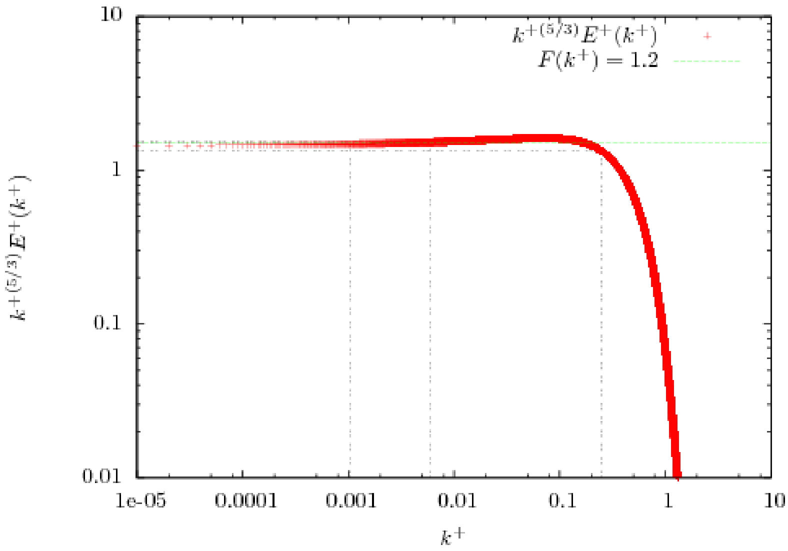

Fig. 25: $k^{+^{(5/3)}} E^{+}(k^{+})$

Table. 7: Wavenumber which departs from inertial subrange within some margin

	k^{+}	$k^{+^{5/3}} E^{+}(k^{+})$
1%	1.04×10^{-3}	1.515
3%	5.87×10^{-3}	1.545
10%	0.2454	1.35

b.

Use the Lin/Hill results above (and anything else you need to assume, of course stating clearly what it is ? Hint: think local equilibrium range) to create a plot of the spectral flux (normalized by the dissipation), $\epsilon_k(k)/\epsilon$. Remember we assumed it to be constant over the inertial range in order to derive the $k^{-5/3}$ result. Is it reasonably constant over the1%, 3% 10% ranges you identified above for the high wavenumber end of the inertial subrange? What is your opinion about using the existence (or non-existence) of a $k^{-5/3}$ range as an indicator of the existence of the spectral gap (at least at the high wavenumber end).

A.

The spectral flux normalized by dissipation is given by the question 5 in the derivation part,

$$\frac{\epsilon_k}{\epsilon} = \exp\left\{-2\alpha_{Kol}\left[\frac{(k\eta_{Kol})^{4/3}}{4/3} + \frac{(k\eta_{Kol})^2}{2}\right]\right\} \tag{3.25}$$

shown in Fig.26. The 1% and 3% range is relatively reasonably but 10% range is diverted from the constant range. In the inertial range the difference of the transfer flux is constant. This constant range of the transfer flux is more sensitive than the flatness of $k^{+^{-5/3}}$ range. For that reason I don't agree with using the $k^{+^{-5/3}}$ range as an indicator of the existence of the spectral gap.

(c).

Using your L/H spectra above, make a plot of $2k^{+^2}E^{+}$, the integral under which is the normalized (in Kolmogorov variables) dissipation. Prove this. Where is the peak? At what wavenumbers (relative to the inverse of the Kolmogorov microscale) does most of the dissipation take place.

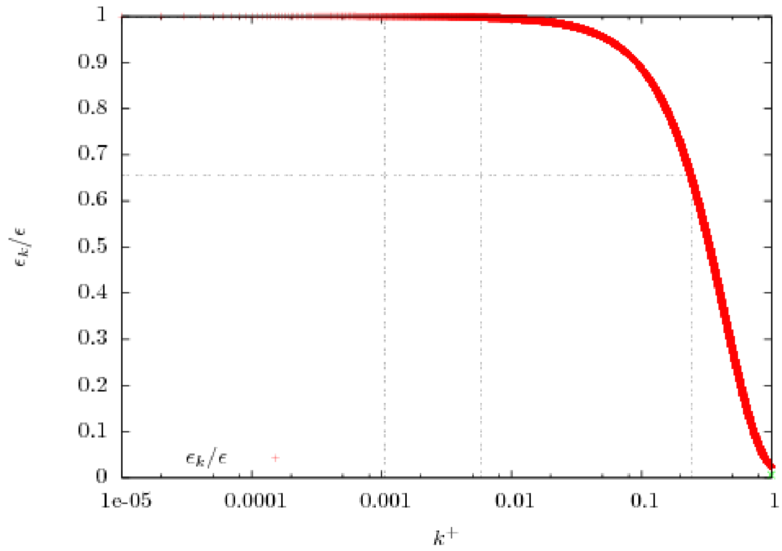

Fig. 26: The normalized spectral flux

A.

The normalized dissipation is given by:

$$\int_0^\infty 2k^{+^2} E^+(k^+) dk^+ = \frac{\int_0^\infty 2k^2 E(k) dk}{\epsilon} \tag{3.26}$$

If the left hand side of this equation is equal to 1 or the numerator of the right hand side is exactly dissipation rate ϵ, it means that the integral of the given equation is the normalized equation. The integral of the given equation is

$$\int_0^\infty 2k^{+^2} E^+(k^+) dk^+ \tag{3.27}$$

Substituting Lin/Hill energy spectrum into this,

$$\begin{aligned}
&= \int_0^\infty 2k^{+^2} \alpha_K k^{+^{-5/3}} [1 + k^{+^{2/3}}] \exp\{-\alpha_K[(3/2)k^{+^{-4/3}} + k^{+^{-2}}]\} dk^+ \\
&= \int_0^\infty 2\alpha_K k^{+^{1/3}} [1 + k^{+^{2/3}}] \exp\{-\alpha_K[(3/2)k^{+^{-4/3}} + k^{+^{-2}}]\} dk^+ \\
&= \int_0^\infty 2\alpha_K [k^{+^{1/3}} + k^+] \exp\{-\alpha_K[(3/2)k^{+^{-4/3}} + k^{+^{-2}}]\} dk^+ \\
&= -\int_0^\infty \left\{\exp\{-\alpha_K[(3/2)k^{+^{-4/3}} + k^{+^{-2}}]\}\right\}' dk^+ \\
&= -\left[\exp\{-\alpha_K[(3/2)k^{+^{-4/3}} + k^{+^{-2}}]\}\right]_0^\infty \\
&= -[0 - 1] \\
&= 1
\end{aligned}$$

Q.E.D.

The peak is at $k^+ = 0.23$, where the most of the dissipation take place.

d.

Now use the isotropic spectral relations to compute what $F_{1,1}^{(1)}$ looks like in Kolmogorov variables assuming $E(k)$ to be given by the Lin-Hill spectrum. (You will have to do this numerically.) Plot both this and it multiplied by $k_1^{+^2}$, and note where the peak is located. How does this location compare to where the maximum in $k^2 E$ was located?

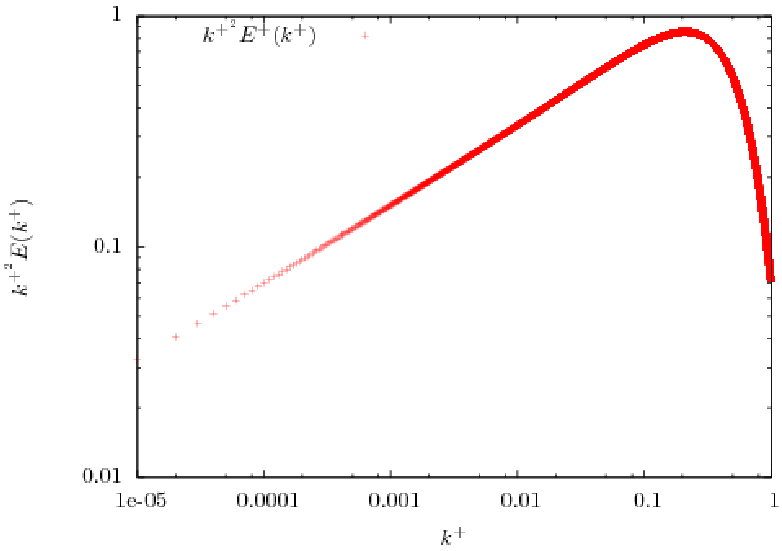

Fig. 27: $k_1^{+^2}E^+(k^+)$

A.

The isotropic spectrum relation is given by:

$$F_{1,1}^{(1)}(k_1) = \frac{1}{2}\int_{k_1}^{\infty} \frac{E(k)}{k}\left[1-\left(\frac{k_1}{k}\right)^2\right]dk \tag{3.28}$$

Using the relation between $E(k)$ and $E(k^+)$ which is given by:

$$E(k^+) = \frac{E(k)}{\nu^{5/4}\epsilon^{1/4}} \tag{3.29}$$

Eq.3.28 becomes

$$\frac{F_{1,1}^{(1)}(k_1)}{\nu^{5/4}\epsilon^{1/4}} = \frac{1}{2}\int_{k_1}^{\infty} \frac{E(k^+)}{k}\left[1-\left(\frac{k_1}{k}\right)^2\right]dk \tag{3.30}$$

Changing the variable from k to k^+ using $k^+ = k\eta_{Kol}$,

$$\frac{F_{1,1}^{(1)}(k_1)}{\nu^{5/4}\epsilon^{1/4}} = \frac{1}{2}\int_{k_1\eta_{Kol}}^{\infty} \frac{E(k^+)}{k^+/\eta_{Kol}}\left[1-\left(\frac{k_1}{k^+/\eta_{Kol}}\right)^2\right]\frac{dk^+}{\eta_{Kol}} \tag{3.31}$$

where $dk^+ = \eta_{Kol}dk$, the integral region becomes from $k : [k_1 \ \ \infty]$ to $k^+ : [k_1\eta_{Kol} \ \ \infty]$. Replacing $\eta_{Kol}k_1$ to k_1^+,

$$\frac{F_{1,1}^{(1)}(k_1)}{\nu^{5/4}\epsilon^{1/4}} = \frac{1}{2}\int_{k_1^+}^{\infty} \frac{E(k^+)}{k^+}\left[1-\left(\frac{k_1^+}{k^+}\right)^2\right]dk^+ \tag{3.32}$$

To solve this numerically, I use the rectangle method which is given by:

$$\frac{F_{1,1}^{(1)}(k_1)}{\nu^{5/4}\epsilon^{1/4}} = \frac{1}{2}\sum_{k^+=k_1^+}^{\infty}\left\{\frac{E(k^+)}{k^+}\left[1-\left(\frac{k_1^+}{k^+}\right)^2\right]\delta k^+\right\} \tag{3.33}$$

The plot is shown in Fig.28. I also plot it multiplied by $k_1^{+^2}$ shown in Fig.29.

The plot of $k^{+^2}E(k^+)$ is shown in Fig.27. The peak is sharper, and the place of the peak is at a bit higher wavenumber than the one of $k^{+^2}F_{1,1}^{(1)}$.

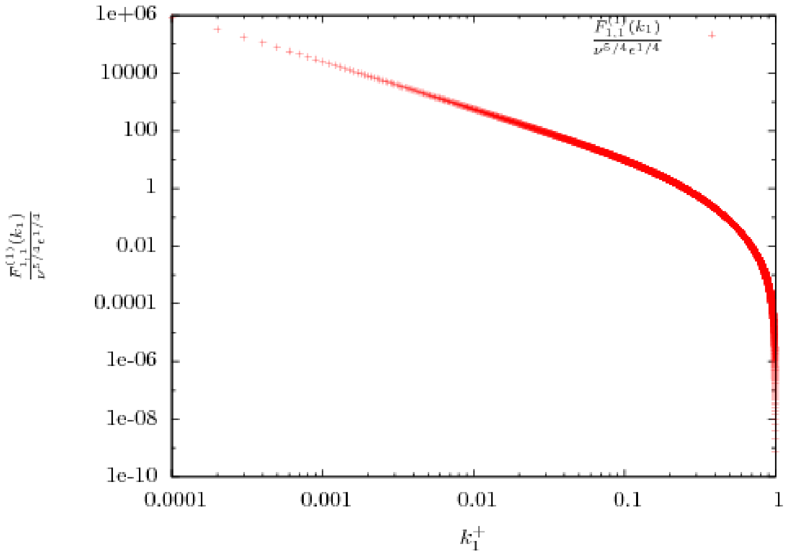

Fig. 28: $\frac{F_{1,1}^{(1)}(k_1)}{\nu^{5/4}\epsilon^{1/4}}$

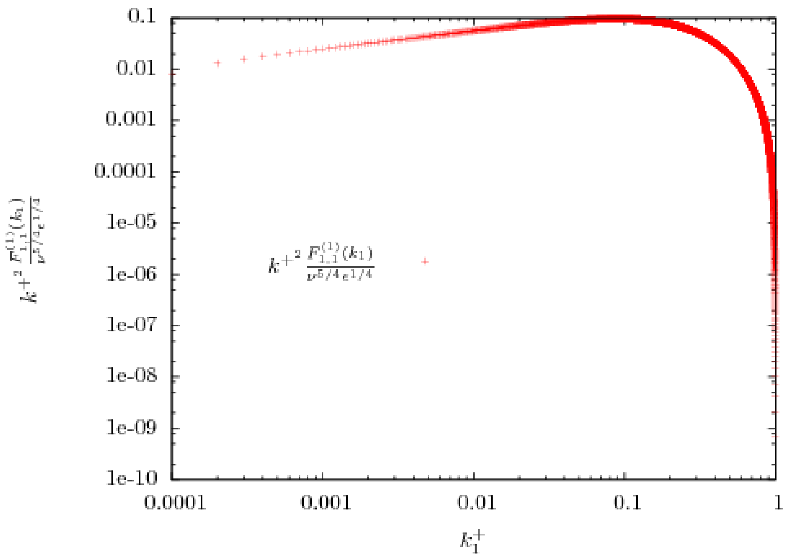

Fig. 29: $k_1^{+^2}\frac{F_{1,1}^{(1)}(k_1)}{\nu^{5/4}\epsilon^{1/4}}$

e.

Assuming a probe can resolve to approximately half its length, how small do you think a problem has to (in Kolmogorov variables) to resolve 99 % of the dissipation? What do you think you would measure for the dissipation (approximately) if your probe were 10 η_{Kol} in length and filtered all scales smaller than this?

A.

The normalized dissipation is given by:

$$\int_0^\infty 2k^{+^2} E^+(k^+) dk^+ \tag{3.34}$$

Using the rectangle method, the equation becomes

$$\sum_0^\infty 2k^{+^2} E^+(k^+) \delta k^+ \tag{3.35}$$

99 % of dissipation implies the sum of the dissipation spectrum from 0 to some wavenumber where the accumulation reaches to 0.99 since the value of the normalized dissipation is exactly 1. The wavenumber at 99 % of the dissipation is obtain as $k^+_{99\%} = 1.136$. The length l corresponding to the this wavenumber is written by:

$$\begin{aligned} l &= \frac{2\pi}{k} \\ &= \frac{2\pi}{1.136}\eta_{Kol} \\ &= 5.53\eta_{Kol} \end{aligned}$$

When the length of our probe is $10\eta_{Kol}$, the scale which can be resolved is $\frac{1}{2}10\eta_{Kol} = 5\eta_{Kol}$. Since $5\eta_{Kol} < 5.53\eta_{Kol}$, the probe is able to resolve smaller scales than the scale corresponding 99 % of the dissipation. Therefore we can measure the dissipation with this probe.

4.

Now create a composite spectrum. First change the $k\eta_{Kol}$ in the L/H spectrum to kL variable using a given value of L/η_K; i.e $k^+ = k\eta_K = KL \times \eta_{Kol}/L = \bar{k} \times \eta_{Kol}/L$. Now multiply the two spectra together and divide by the common part, which is just $\alpha_K \bar{k}^{-5/3}$. This is your composite spectrum valid at all wavenumbers.

(a)

Plot it for values of $L/\eta_{Kol} = 10000, 3000, 1000, 300$ and 30.

A.

The Lin/Hill energy spectrum is given by:

$$E^+(k^+) = \alpha_K k^{+^{-5/3}} [1 + k^{+^{2/3}}] \exp\{-\alpha_K[(3/2)k^{+^{-4/3}} + k^{+^{-2}}]\} \tag{3.36}$$

Now I replace k^+ to $\bar{k} \times \eta_{Kol}/L$. Eq.3.36 becomes

$$E\left(\bar{k}\frac{\eta_{Kol}}{L}\right) = \alpha\bar{k}^{-5/3}\left(\frac{\eta_{Kol}}{L}\right)^{-5/3}\left[1 + \bar{k}^{2/3}\left(\frac{\eta_{Kol}}{L}\right)^{2/3}\right]\exp\left\{-2\alpha_{Kol}\left[\frac{3}{2}\bar{k}^{4/3}\left(\frac{\eta_{Kol}}{L}\right)^{4/3} + \bar{k}^2\left(\frac{\eta_{Kol}}{L}\right)^2\right]\right\} \tag{3.37}$$

While the vk/H energy spectrum is given by:

$$\bar{E}(\bar{k}) = \frac{E}{u^2 L} = \frac{C_p \bar{k}^4}{\{1 + (\bar{k}/\bar{k}_e)^2\}^{17/6}} \tag{3.38}$$

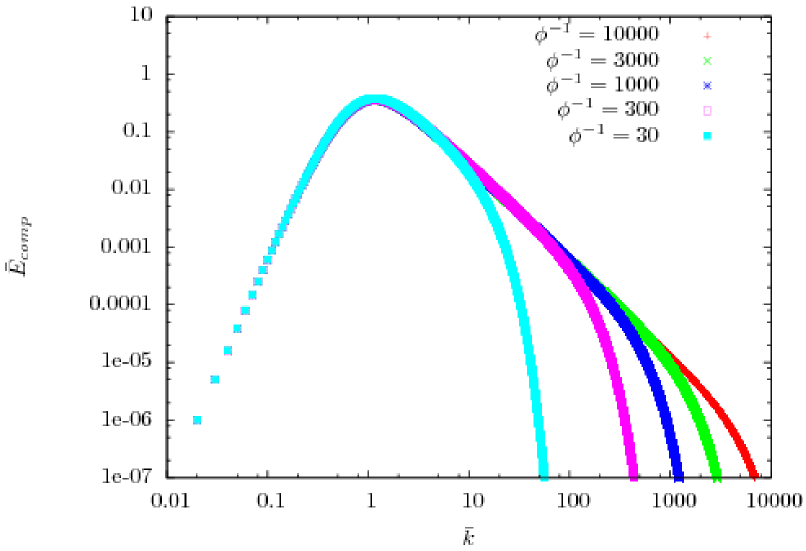

Fig. 30: Composite energy spectrum at the given ϕ

Hence the composite energy spectrum is obtained by the product of the spectra divided by the common part[10];

$$\begin{aligned}
\bar{E}_{comp} &= \frac{\bar{E}_{L/H}\bar{E}_{vK/H}}{\alpha_K k^{+-5/3}} \\
&= \frac{C_p \bar{k}^4}{\{1+(\bar{k}/\bar{k}_e)^2\}^{17/6}} \left[1 + \bar{k}^{2/3}\left(\frac{\eta_{Kol}}{L}\right)^{2/3}\right] \\
&\quad \times \exp\left\{-2\alpha_{Kol}\left[\frac{3}{2}\bar{k}^{4/3}\left(\frac{\eta_{Kol}}{L}\right)^{4/3} + \bar{k}^2\left(\frac{\eta_{Kol}}{L}\right)^2\right]\right\} \\
\bar{E}_{comp} &= \frac{C_p \bar{k}^4}{\{1+(\bar{k}/\bar{k}_e)^2\}^{17/6}} \left(1 + \bar{k}^{2/3}\phi^{2/3}\right) \exp\left\{-2\alpha_{Kol}\left[\frac{3}{2}\bar{k}^{4/3}\phi^{4/3} + \bar{k}^2\phi^2\right]\right\} \qquad (3.39) \\
\text{where} \quad \phi &= \frac{\eta_{Kol}}{L}
\end{aligned}$$

All the plots are shown in Fig.30

(b)

Now plot $\bar{k}^{5/3}\bar{E}$ (on the same plot if possible) for these values. Note and comment on what happens to the inertial subrange, and how the Reynolds number would affect any assumptions about its existence.

A.

The plots are shown in Fig.31. The inverse of ϕ is L/η_{Kol} which means how large Reynolds number is. The more Reynolds number increases, the more inertial range expands. Hence the existence of inertial range is affected by how high Reynolds number is.

[10] In the sentence of the question it says the common part is $\alpha_K \bar{k}^{-5/3}$ but actually it isn't true. The correct common part should be $\alpha_K k^{+-5/3}$, and it gives us the correct graph of the spectra (Fig. 30).

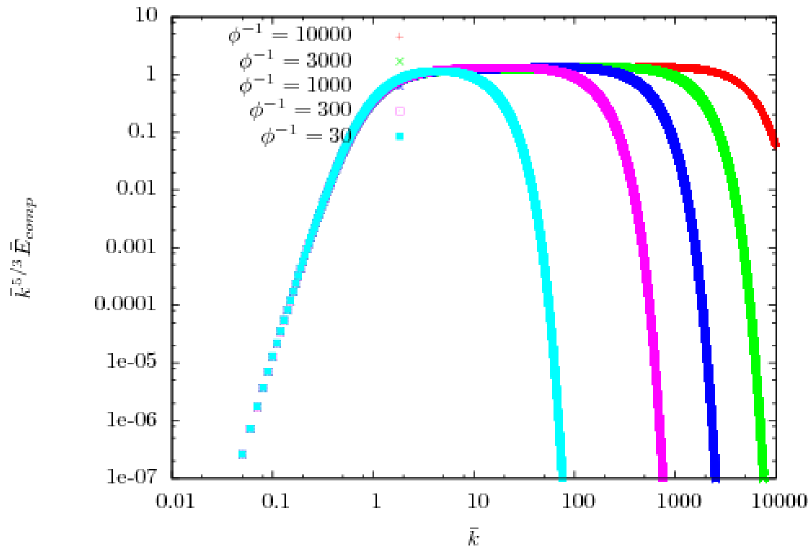

Fig. 31: $\bar{k}^{5/3}\bar{E}_{comp}$ at the given ϕ

4 PartIV :Application

5.

The dissipation spectrum (assuming the turbulence to be nearly isotropic) is given by:

$$\epsilon = 2\nu \int_0^\infty k^2 E(k) dk \tag{4.1}$$

or

$$\epsilon = 30\nu \int_0^\infty k_1^2 F_{1,1}^{(1)}(k_1) dk_1 \tag{4.2}$$

(a)

Use the isotropic relation for the three-dimensional spectrum to prove equation 6 is in fact the dissipation.

A.

The dissipation is expressed as

$$\epsilon = 2\nu \int_0^\infty k^2 E(k) dk \tag{4.3}$$

The isotropic relation between $E(k)$ and $F_{1,1}^{(1)}$ is given by:

$$E(k) = k^3 \frac{d}{dk}\left[\frac{1}{k}\frac{dF_{1,1}^{(1)}}{dk}\right] dk \tag{4.4}$$

Plugging it into the dissipation equation,

$$\begin{aligned}
\epsilon &= 2\nu \int_0^\infty k^5 \frac{d}{dk}\left[\frac{1}{k}\frac{dF_{1,1}^{(1)}}{dk}\right] dk \\
&= 2\nu \left[k^5 \frac{1}{k}\frac{dF_{1,1}^{(1)}}{dk}\right]_0^\infty - 2\nu \int_0^\infty 5k^4 \frac{1}{k}\frac{dF_{1,1}^{(1)}}{dk} dk \\
&= -10\nu \int_0^\infty k^3 \frac{dF_{1,1}^{(1)}}{dk} dk \\
&= -10\nu \left[k^3 F_{1,1}^{(1)}\right]_0^\infty + 30\nu \int_0^\infty k^2 F_{1,1}^{(1)}(k) dk \\
&= 30\nu \int_0^\infty k^2 F_{1,1}^{(1)}(k) dk
\end{aligned} \tag{4.5}$$

(b)

Use the spectrum you obtained in problem 1 to make a plot of the velocity derivative spectrum, $k_1^2 F_{1,1}^{(1)}$ (linear-linear ok). What velocity derivative is this the spectrum of? What fact about isotropy do you need to obtain the equations above?

A.

The plot of $k_1^2 F_{1,1}^{(1)}$ is shown in Fig.32.

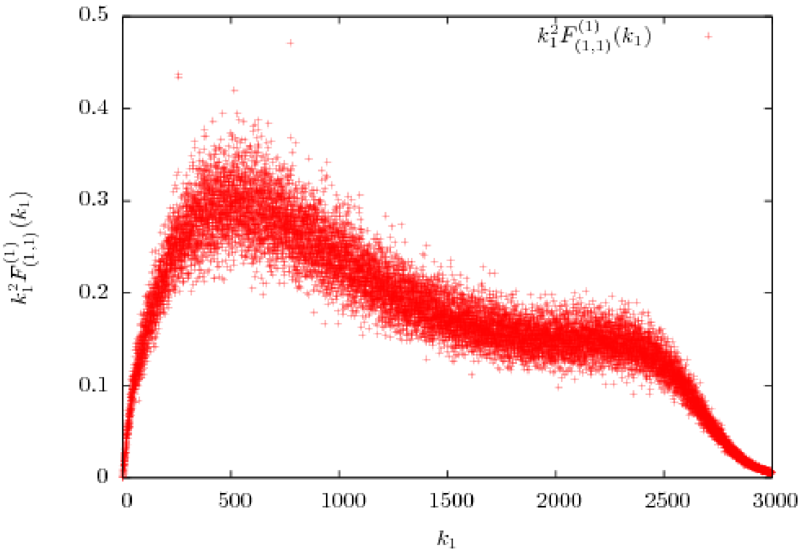

Fig. 32: The plot of $k_1^2 F^{(1)}_{1,1}$

The velocity derivative can be expressed as

$$
\begin{aligned}
k_1^2 F^{(1)}_{1,1}(k_1) &= \frac{k_1^2}{2\pi}\int_{-\infty}^{\infty} B(r,0,0)e^{-ik_1 r}dr \qquad (4.6)\\
&= \frac{ik_1}{2\pi}\int_{-\infty}^{\infty} B(r,0,0)\left\{e^{-ik_1 r}\right\}' dr\\
&= \frac{ik_1}{2\pi}\left[B(r,0,0)e^{-ik_1 r}\right]_{-\infty}^{\infty} - \frac{ik_1}{2\pi}\int_{\infty}^{\infty}\frac{\partial}{\partial r}B(r,0,0)e^{-ik_1 r}dr\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\partial}{\partial r}B(r,0,0)\left\{e^{-ik_1 r}\right\}' dr\\
&= \frac{1}{2\pi}\left[\frac{\partial}{\partial r}B(r,0,0)e^{-ik_1 r}\right]_{-\infty}^{\infty} - \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\partial^2}{\partial r^2}B(r,0,0)e^{-ik_1 r}dr\\
&= -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\partial^2}{\partial r^2}B(r,0,0)e^{-ik_1 r}dr \qquad (4.7)
\end{aligned}
$$

where $B(r,0,0)$ is given by:

$$B(r,0,0) = \langle u(x+r)u(x)\rangle \qquad (4.8)$$

and

$$x + r = x' \qquad (4.9)$$

For fixed x',

$$\frac{\partial}{\partial r} = -\frac{\partial}{\partial x} \qquad (4.10)$$

and for fixed x,

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial x'} \qquad (4.11)$$

The derivative of $B(r,0,0)$ becomes

$$
\begin{aligned}
\frac{\partial^2}{\partial r^2}B(r,0,0) &= \frac{\partial^2}{\partial r^2}\langle u(x+r)u(x)\rangle\\
&= -\frac{\partial}{\partial x}\frac{\partial}{\partial x'}\langle u(x')u(x)\rangle\\
&= -\left\langle \frac{\partial}{\partial x'}u(x')\frac{\partial}{\partial x}u(x)\right\rangle \qquad (4.12)
\end{aligned}
$$

Hence I guess the first order derivative is important. Now considering the Fourier transfer of the first order derivative of the velocity,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\partial}{\partial x}u(x)e^{-ik_1x}dx = -ik_1\hat{u}(k_1) \tag{4.13}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\partial}{\partial x'}u(x')e^{-ik_1'x'}dx' = -ik_1'\hat{u}(k_1') \tag{4.14}$$

where $\hat{u}(k_1')$ is the complex conjugate of $\hat{u}(k_1)$. An ensemble average of the product of this two yields:

$$-k_1k_1'\langle\hat{u}(k_1)\hat{u}(k_1')\rangle \tag{4.15}$$

Now using the fact[11] we learned:

$$\langle\hat{u}(k_1)\hat{u}(k_1')\rangle = F_{1,1}(k_1)\delta(k_1-k_1') \tag{4.16}$$

Then,

$$= -k_1k_1'F_{1,1}(k_1)\delta(k_1-k_1') \tag{4.17}$$

where

$$k_1'\delta(k_1-k_1') = -k_1 \tag{4.18}$$

because $\delta(k_1-k_1') = 1$ when $k_1' = -k_1$ otherwise $\delta(k_1-k_1') = 0$. Finally,

$$\begin{aligned} & -k_1k_1'F_{1,1}(k_1)\delta(k_1-k_1') \\ = \; & k_1^2F_{1,1}(k_1) \end{aligned} \tag{4.19}$$

Therefore the velocity derivative is $\frac{\partial}{\partial x}u(x)$.

(c)

Now use your one-dimensional spectrum computed in problem 1 to make an estimate of ϵ, and compare it to your estimate obtained from the assumed universality of the inertial subrange. Can you explain any differences? (Hint: In the boundary layers experiment above, the wire length, l_w, was estimated to be about 2 mm long. How does this compare to the Kolmogorov microscale. Where would you expect the peak in the dissipation spectrum to lie relative to it? Relative to the wire cut-off?)

A.

The dissipation rate ϵ is given by:

$$\begin{aligned} \epsilon_{spectrum} &= 30\nu\int_0^{\infty}k_1^2F_{1,1}^{(1)}(k_1)dk_1 \\ &= 0.240\ [\mathrm{m}^2/\mathrm{s}^3] \end{aligned} \tag{4.20}$$

where I assume the working fluid is the air in the standard condition, $\nu = 1.5\times10^{-5}[\mathrm{m}^2/\mathrm{s}]$. The dissipation rate estimated from the inertial range is $\epsilon_{Inertial\ range} = 0.0838\ [\mathrm{m}^2/\mathrm{s}^3]$. Two Kolmogorov scale for each method are obtained by the definition;

$$\begin{aligned} \eta_{spectrum} &= \left(\frac{\nu^3}{\epsilon_{spectrum}}\right)^{\frac{1}{4}} \\ &= 3.44\times10^{-4}\ [m] \\ \eta_{Inertialrange} &= \left(\frac{\nu^3}{\epsilon_{Inertialrange}}\right)^{\frac{1}{4}} \\ &= 4.48\times10^{-4}\ [m] \end{aligned}$$

[11] Your note, page 204

$\eta_{spectrum}$ comes from the isotropic relation, while $\eta_{inertial\ range}$ comes from the -5/3 range. $-5/3$ range doesn't assume the isotropy but only the homogeneity.

The peak is expected to locate at larger scale than these values. Because the most dissipative scale is bigger than the Kolmogorov scale.

The wire cut-off scale is $2 \times 2 \times 10^{-3}$[m], so we can resolve the smaller scales than 4×10^{-3} [m] with this probe. The peak is also expected to locate at larger scale than this value, because otherwise we cannot resolve the most dissipative scale during the measurement.

(d)

By using your knowledge about how a hot-wire averages along its length, your knowledge of isotropic turbulence, and the Lin/Hill spectral model, compute the "measured" one-dimensional spectrum you would obtain with a wire of this length, and compare it to the " unfiltered " spectra. E.G. Compute the "measured spectrum" from:

$$F^{1m}_{1,1}(k_1) = \iint_{-\infty}^{\infty} F_{1,1}(k_1, k_2, k_3) W_L(k_2) dk_2 d_3 \tag{4.21}$$

where

$$W_L(k_2) = |F.T.\{w_L(x_2)\}|^2 \tag{4.22}$$

and

$$w_L(x_2) = 1, |x_2| \leq l_w/2 \tag{4.23}$$
$$= 0, |x_2| > l_w/2 \tag{4.24}$$

A.

The Fourier transform of $w_L x_2$ is given by:

$$\begin{aligned}\mathcal{F}\{w_L(x_2)\} &= \int_{-\frac{l_w}{2}}^{\frac{l_w}{2}} e^{-ik_2x_2} dx_2 \\ &= \left[\frac{e^{-ik_2x_2}}{-ik_2}\right]_{-\frac{l_w}{2}}^{\frac{l_w}{2}} \\ &= \left[\frac{e^{-ik_2\frac{l_w}{2}} - e^{ik_2\frac{l_w}{2}}}{-ik_2}\right] \\ &= \frac{2}{k_2} \sin k_2 \frac{l_w}{2}\end{aligned} \tag{4.25}$$

$W_L(k_2)$ is given by the square of $\mathcal{F}\{w_L x_2\}$;

$$W_L(k_2) = \frac{4}{k_2^2} \sin^2 k_2 \frac{l_w}{2} \tag{4.26}$$

Considering how to average over the length of the hot wire[12], "filtered" spectrum becomes

$$F^{1m}_{1,1}(k_1) = \iint_{-\infty}^{\infty} \frac{E(k)}{4\pi k^4}[k^2 - k_1^2] \frac{\sin^2 k_2 \frac{l_w}{2}}{\left(\frac{l_w}{2}\right)^2 k_2^2} dk_2 dk_3 \tag{4.27}$$

where $E(k)$ is Lin/Hill spectrum;

$$E^+(k^+) = \frac{E}{\nu^{5/4}\epsilon^{1/4}} = \alpha_K k^{+^{-5/3}} [1 + k^{+^{2/3}}] \exp\{-\alpha_K [(3/2) k^{+^{-4/3}} + k^{+^{-2}}]\} \tag{4.28}$$

[12]J. H. Citriniti, W. K. George, The reduction of spatial aliasing by long hot-wire anemometer probes, Experiments in Fluids, **23**, 217-224, (1997)

When I choose the variable $\sigma = k^2 - k_1^2$, k_2 and k_3 can be expressed as:

$$k_2 = \sigma \cos\theta \tag{4.29}$$

$$k_3 = \sigma \sin\theta \tag{4.30}$$

$$dk_2 dk_3 = \sigma d\sigma d\theta \tag{4.31}$$

Substituting them into Eq.4.27,

$$F_{1,1}^{1m}(k_1) = \int_{\sigma=0}^{\infty}\int_{\theta=0}^{2\pi} \frac{\nu^{5/4}\epsilon^{1/4}E^+(k^+)}{4\pi k^2}\left[1-\frac{k_1^2}{k^2}\right]\left[\frac{\sin\frac{l_w}{2}\sigma\cos\theta}{\frac{l_w}{2}\sigma\cos\theta}\right]^2 \sigma d\sigma d\theta \tag{4.32}$$

For fixed k_1, $\sigma d\sigma = kdk$. Hence the equation becomes

$$F_{1,1}^{1m}(k_1) = \int_{k=k_1}^{\infty}\int_{\theta=0}^{2\pi} \frac{\nu^{5/4}\epsilon^{1/4}E^+(k^+)}{4\pi k^2}\left[1-\frac{k_1^2}{k^2}\right]\left[\frac{\sin\frac{l_w}{2}\sqrt{k^2-k_1^2}\cos\theta}{\frac{l_w}{2}\sqrt{k^2-k_1^2}\cos\theta}\right]^2 kdkd\theta \tag{4.33}$$

Since $k^+ = k\eta_{Kol}$,

$$F_{1,1}^{1m}(k_1) = \int_{k=k_1\eta_{Kol}}^{\infty}\int_{\theta=0}^{2\pi} \frac{\nu^{5/4}\epsilon^{1/4}E^+(k^+)}{4\pi\frac{k^+}{\eta_{Kol}}}\left[1-\frac{k_1^{+^2}}{k^{+^2}}\right]\left[\frac{\sin\frac{l_w^+}{2}\sqrt{k^{+^2}-k_1^{+^2}}\cos\theta}{\frac{l_w^+}{2}\sqrt{k^{+^2}-k_1^{+^2}}\cos\theta}\right]^2 \frac{dk^+}{\eta_{Kol}}d\theta \tag{4.34}$$

Therefore,

$$\frac{F_{1,1}^{1m}(k_1)}{\nu^{5/4}\epsilon^{1/4}} = \int_{k=k_1^+}^{\infty}\int_{\theta=0}^{2\pi} \frac{E^+(k^+)}{4\pi k^+}\left[1-\frac{k_1^{+^2}}{k^{+^2}}\right]\left[\frac{\sin\frac{l_w^+}{2}\sqrt{k^{+^2}-k_1^{+^2}}\cos\theta}{\frac{l_w^+}{2}\sqrt{k^{+^2}-k_1^{+^2}}\cos\theta}\right]^2 dk^+d\theta \tag{4.35}$$

"Unfiltered" spectrum is given by:

$$\frac{F_{1,1}^{1}(k_1)}{\nu^{5/4}\epsilon^{1/4}} = \int_{k_1^+}^{\infty}\frac{E^+(k^+)}{4\pi k^+}\left[1-\frac{k_1^{+^2}}{k^{+^2}}\right]dk^+ \tag{4.36}$$

The differences are the sinc function in filtered spectrum and the integration along θ. When I compute to integrate Eq. (4.35) numerically, I face to the difficulty of the oscillation of $\sin(\cos\theta)$. To avoid this I attempt to change variable as the follows;

$$x = \frac{l_w^+}{2}\sqrt{k^{+^2}-k_1^{+^2}}\cos\theta \tag{4.37}$$

The first order derivative is given by:

$$dx = -\frac{l_w^+}{2}\sqrt{k^{+^2}-k_1^{+^2}}\sin\theta d\theta \tag{4.38}$$

And the integral region of θ changes from $[0{:}2\pi]$ to $\left[-\frac{l_w^+}{2}\sqrt{k^{+^2}-k_1^{+^2}} : \frac{l_w^+}{2}\sqrt{k^{+^2}-k_1^{+^2}}\right]$. Eq.(4.35) becomes

$$\frac{F_{1,1}^{1m}(k_1)}{\nu^{5/4}\epsilon^{1/4}} = \int_{k=k_1^+}^{\infty}\int_{x=-\frac{-l_w^+}{2}\sqrt{k^{+^2}-k_1^{+^2}}}^{\frac{-l_w^+}{2}\sqrt{k^{+^2}-k_1^{+^2}}} \frac{E^+(k^+)}{4\pi k^+}\left[1-\frac{k_1^{+^2}}{k^{+^2}}\right]\left[\frac{\sin x}{x}\right]^2 \frac{1}{\frac{-l_w^+}{2}\sqrt{k^{+^2}-k_1^{+^2}}\sin\theta}dk^+dx \tag{4.39}$$

Now $\sin\theta$ can be written by:

$$\sin\theta = \sin\left\{\arccos\left(\frac{2x}{l_w^+\sqrt{k^{+^2}-k_1^{+^2}}}\right)\right\} \tag{4.40}$$

Finally,

$$\frac{F_{1,1}^{1m}(k_1)}{\nu^{5/4}\epsilon^{1/4}} = \int_{k=k_1^+}^{\infty}\int_{x=-\frac{-l_w^+}{2}\sqrt{k^{+^2}-k_1^{+^2}}}^{\frac{-l_w^+}{2}\sqrt{k^{+^2}-k_1^{+^2}}} \frac{\frac{E^+(k^+)}{4\pi k^+}\left[1-\frac{k_1^{+^2}}{k^{+^2}}\right]\left[\frac{\sin x}{x}\right]^2}{\frac{-l_w^+}{2}\sqrt{k^{+^2}-k_1^{+^2}}\sin\left\{\arccos\left(\frac{2x}{l_w^+\sqrt{k^{+^2}-k_1^{+^2}}}\right)\right\}}dk^+dx \tag{4.41}$$

But I couldn't get a good result. The 'filtered' spectrum is the always same as the 'unfiltered' spectrum like Fig. 28. That means sinc function doesn't affect the spectrum in my program.

(e)

Compare the result with the actual measured spectra above by plotting both the velocity and velocity derivative spectra. How much would you judge the errors caused by the finite wire length to have affected the velocity derivative measurement (and hence the dissipation)? (Hint: integrate the difference.)