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INSIGHT INTO THE DYNAMICS OF COHERENT STRUCTURES FROM A PROPER ORTHOGONAL DECOMPOSITION

by

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INTRODUCTION

The Danish cartoonist-philosopher, Piet Hein, in one of his more famous grooks, tells of the scientist who has discovered a well which is the source of truth. After carefully filling his beaker from the well, he examines the liquid closely, then pronounces his conclusion: Truth is cylindrical. A similar insight into the nature of human investigators is contained in the poem by John Godfrey Saxe in which six blind men discover an elephant. The first falls against the side and proclaims the elephant to be like a wall, and the second feeling a tusk suggests the elephant is more like a spear. The third grasps the trunk and argues that the elephant is much like a snake, while the fourth embracing a knee argues that it is really more like a tree. The fifth feeling an ear believes the elephant to be more like a fan, and the sixth seizing the tail proclaims that the elephant is clearly more like a rope.

This is a paper about coherent structures, and one of the tools we use to study them -- the proper orthogonal decomposition. Like the scientist and the blind men in the illustrations above, one's beliefs about coherent structures are largely shaped by his perspective. And like them, there is a tendency for those who study coherent structures to want the answers to be simple.

The purpose of this paper is to provide some insight into the particular perspective of the proper orthogonal decomposition. The focus will be on what the orthogonal decomposition can tell us about the mathematical representation of coherent stuctures which is necessary to capture their essential kinematics. Without such a mathematical description, coherent stuctures are likely to remain forever no more than a curiosity for the fluid mechanics community, with little possibility of influencing the manner in which turbulence models are built. The hope is that by finding a way to represent stuctures and events which is true to nature, we will open at long last the possibility of exploring the importance of coherent structures for the dynamics of turbulent flow.

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There will be no attempt in this paper to provide a complete history of orthogonal decomposition techniques, nor even to list the ever-increasing attempts to utilize them. Rather, the focus will be on 'What is a proper orthogonal decomposition in the context of turbulence?' and on 'What does it have to offer us in the study of coherent structures?'. The hope is that this presentation will lead to a better understanding of what has already been done, and to a broader discussion of where we should go from here.

THE PROPER ORTHOGONAL DECOMPOSITION

Like most good ideas, the proper othogonal decomposition we discuss here has more one who can lay claim to having invented it. Oceanographers and meteorologists, who have used it for the study of various geophysical phenomena, neatly by-pass the question of origin and refer to their results as Empirical Orthogonal Functions (EOF's), bristling whenever anyone suggests using another term. The mathematicians are equally adamant about referring to it as a Karhunen-Loeve expansion. The turbulence community (or at least a part of it) are happy to refer to it as Lumley's Orthogonal Decomposition. All of these choices can be defended. Certainly it is a Karhunen-Loeve expansion, and the eigenfunctions obtained are empirical. And the recognition of its applicability to turbulence (and perhaps even its generalization to multi-dimensions) is due to Lumley (1967). The viewpoint of this paper has been very strongly influenced by the latter, although some of the interpretations offered here may differ from those originally proposed. Nonetheless, for reasons of efficiency and to offend no one (or at least all equally) we shall in the remainder of this paper simply use the term Orthogonal Decomposition (or OD).

The basic idea behind the Orthogonal Decomposition (OD) is that one tries to optimally represent a random field by a set of deterministic functions which are in turn determined by the field itself. This is quite unlike the more common situation where one chooses a set of orthogonal functions (like the harmonic ones of Fourier analysis) and then seeks the coefficients necessary to represent the field. For the OD, both the functions and the coefficients arise from the statistical properties of the random field itself.

The key word for determining the orthogonal functions is "optimal". For the OD discussed here, "optimal" means that the functions are to be determined so that the most possible energy is contained in the lowest order term, then the most possible of the remaining energy in the next, and so on. This can be accomplished for a random field $u_i(\underline{x},t)$ by finding a deterministic field $\phi_i(\underline{x},t)$ for which the mean square projection of ϕ_i on u_i is maximal. If we denote this projection by α where

$$\alpha = u_i \ \phi_i = \underline{u} \cdot \underline{\phi} \tag{1}$$

then we must maximize

$$<|\alpha|^2> = <|u_i\phi_i|^2> \tag{2}$$

where < > denotes the ensemble average. From the calculus of variations, ϕ_i can be shown to be determined by the following integral equation,

$$\iiint_{\text{all space}} R_{i,j}(\cdot,\cdot')\phi_j(\cdot')d(\cdot') = <|\alpha|^2>\phi_i(\cdot) \tag{3}$$

where

$$(\cdot) = (\underline{x},t)$$

and

$$(\cdot') = (\underline{x}',t')$$

and where $R_{i,j}$ represents the cross-correlation defined by

$$R_{i,j}(\cdot,\cdot') = <u_i(\cdot)u_j(\cdot')> \tag{3}$$

Thus the problem of optimally determining the deterministic function ϕ_i has been reduced to an eigenvalue problem with the kernel given by the cross-correlation of the random vector field.

Since part of our purpose here is perspective, it is worth pausing for a moment to note that there is no longer room for judgement or subjectivity on the part of the theoretician or experimentalist. By deciding at the beginning that the functions we sought should optimally represent the field in a mean square sense (equation2), the functions have been completely determined. It actually matters not whether the field represents velocity, vorticity, pressure, or temperature, the integral of equation (3) will simply have as its kernel the appropriate cross-correlation. Thus all of our subjectivity has been lumped into equation (2) and the choice of principle behind it. It might be interesting to explore the consequences of alternative choices for maximization.

The solutions to equation (2) are very strongly dependent on the kind of field being discussed. If the field is of finite total energy, then the solutions to equation

(3) are governed by the Hilbert-Schmidt theory of linear integral equations from which it follows that:

(i) There exists a denumerable set of eigenvalues, λ_n, satisfying equation (3), and $\lambda_1 > \lambda_2 > \lambda_3$, etc.

(ii) All of the eigenvalues are real.

(iii) The eigenfunctions are orthogonal and can be normalized so that

$$\iiint_{\text{Domain}} \phi_i^{(m)}(\cdot)\,\phi_i^{(n)}(\cdot)\,d(\cdot)) = \delta_{mn} \tag{4}$$

(iv) The cross-correlation can be represented by them as

$$R_{i,j}(\cdot,\cdot') = \sum_n \lambda_n \phi_i^{(n)}(\cdot)\ \ \phi_i^{(n)*}(\cdot) \tag{5}$$

(v) Any realization of the random field can be represented as

$$u_i(\cdot) = \sum_n a_n\ \phi_i^{(n)}(\cdot) \tag{6}$$

where the coefficients are determined for each realization of the field by

$$a_n = \iint_{\text{Domain}} u_i(\cdot)\,\phi_i^{(n)}(\cdot)\,d(\cdot) \tag{7}$$

and where

$$\langle a_n a_m^* \rangle = \lambda_n\ \delta_{mn} \tag{8}$$

A corollary of equations (4) and (5) is that the total energy in the field can be obtained as

$$\iint_{\text{Domain}} \langle u_i u_i \rangle\ d(\cdot) = \sum_n \lambda_n \tag{9}$$

Thus, as required, the energy is recovered in an optimal manner by our eigenfunctions and eigenvalues.

Before proceeding further, it is appropriate to ask what this decomposition, however elegant and efficient, has to do with coherent structures? The answer to this question is somewhat complicated by the fact that there exists no definition of a coherent structure with any dynamical or mathematical significance. The situation was aptly described by Lumley (1982) who, drawing from a U.S. Supreme Court decision on pornography, pointed out that like pornography, coherent structures are hard to define, but you know one when you see it. Less facetiously, Lumley (1967) did suggest that the "large eddy" of a turbulent flow (coherent structures had not been invented then) be identified as the lowest order eigenfunction resulting from an orthogonal decomposition of the type outlined above.

Certainly it would appear, to this point at least, that the eigenfunctions determined by equation (3) have many of the desired properties we would associate with coherent structures. They certainly represent coherent features of the flow, and are even deterministic. Unfortunately, as will be seen later, a simple physical representation of the flow field resulting from them is not always possible. Moreover, it will be also be seen that there is some evidence that a given structure is represented by different order eigenfunctions at different points in its life-cycle.

HOMOGENEOUS AND STATIONARY RANDOM FIELDS

The solution to the integral of equation (3) becomes considerably more complicated if the field is not of finite extent, as is the case if it is stationary (temporally) or homogeneous (spatially) in one or more coordinates. In such cases, the eigenfunctions in these coordinates reduce to the harmonic ones and the coefficients are simply the Fourier coefficients. This would not at first sight appear to be a complication, since the eigenfunctions do not have to be determined, but are known a priori. In fact, because the eigenfunctions are known, the decomposition can be performed on these directions immediately without solving an integral equation. There are several equivalent ways of doing this, but probably the simplest is to Fourier transform the random field over the homogeneous and stationary variables, then solve the integral equation for this transformed field to find the optimal orthogonal functions for the remaining directions.

For example, suppose the random field to be stationary in time and homogeneous in the x_1 coordinate. Then the Fourier transform of the field is given by

$$\hat{u}_i(k_1,\omega;x_2,x_3) = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} u_i(x_1,x_2,x_3,t)\ e^{-i(k_1x + \omega t)}\ dx_1dt \qquad (10)$$

We now seek a deterministic function $\psi_i(x_2,x_3;k_1,\omega)$ whose mean square projection on the transformed random field is a maximal. As before, we are led to an integral equation, but this time given by

$$\iint_{\text{Domain}} \Phi_{i,j}(x_2',x_2,x_3',x_3;)\ \psi_j(x_2,x_3;k_1,\omega)\ dx_2 dx_3 = \lambda(k_1,\omega)\ \psi_i(x_2,x_3;k_1,\omega) \qquad (11)$$

$\Phi_{i,j}$ is the cross-spectrum defined as the Fourier transform of the cross-correlation over the appropriate variables, i.e.

$$\Phi_{i,j}(x_2',x_2,x_3',x_3;k_1,\omega) = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} R_{i,j}(x_1'-x_1,x_2',x_2,x_3',x_3,t'-t) \cdot e^{-i[k_1(x_1'-x_1)+\omega(t'-t)]}\ d(x_1'-x_1)\,d(t'-t) \qquad (12)$$

where the use of $x_1'-x_1$ and $t'-t$ reflects the fact that the field is homogeneous in the 1-direction and stationary in time.

Now, instead of the single eigenvalue problem of equation (3), we have an eigenvalue problem for each frequency and/or wavenumber combination. Each of the eigenvalues of equation (11) has now become an eigenspectrum $\lambda(k_1,\omega)$. While this may present some difficulties in managing the data, they are more than compensated for by the reduced information required about the cross-correlation because of the homogeniety and stationarity. (In our example, the number of variables has been reduced from 8 to 6.)

As before, we can reconstruct the decomposed Fourier transformed field, in this case by

$$\hat{u}_i(k_1,\omega;x_2,x_3) = \sum_n a_n(k_1,{}_1w)\ \psi_i^*(n)(x_2,x_3;k_1,\omega) \qquad (13)$$

where the coefficients are determined by

$$a_n(k_1,\omega) = \frac{1}{(2\pi)^2} \iint_{\text{Domain}} \hat{u}_i(k_1,\omega;x_2,x_3) \ \psi_i^{(n)}(x_2,x_3;k_1,\omega) \ dx_2 dx_3 \qquad (14)$$

and satisfy

$$\langle a_n(k_1,\omega) \ a_m^*(k_1,\omega)\rangle = \lambda_n(k_1,\omega) \ \delta_{mn} \qquad (15)$$

Equation (13) can in turn be inverse Fourier transformed to obtain the original field as

$$u_i(x_1,x_2,x_3) = \sum_n \left\{ \iint_{-\infty}^{\infty} a_n(k_1,\omega)\psi_i^{(n)}(x_2,x_3;k_1,\omega) \ e^{i(k_1x_1+\omega t)} dk_1 d\omega \right\} \qquad (16)$$

Thus the original field has been completely represented in terms of the eigenfunctions and their coefficients, and as before, all of its statistics are intact.

The complications alluded to above then do not arise from a short-coming of the decomposition itself, since we have still optimally decomposed the random field. Nor do they arise from the reconstruction of the field, since again the original field is faithfully recovered. The problems occur when we try to reconstruct a physical picture of an individual eddy. As for the completely inhomogeneous problem above, it is straightforward to identify the spatial variation of each mode in the x_1 and t coordinates since they are simple Fourier modes. It is for the inhomogeneous directions, however, that we encounter difficulties because the eigenfunctions are continuous functions of wavenumber and frequency. Thus, we can determine the spatial dependence in the inhomogeneous directions only by first adding up all the Fourier modes to determine the appropiate coefficients according to the recipe of equation (16). Unfortunately, these coefficients are both complex and random, thus the amount of each eigenfunction to be added varies, not only from ensemble to ensemble, but from time to time and place to place within a given ensemble. Thus without additional assumptions, there is no one way to add up the pieces to find out what a "typical" coherent structure looks like.

This inability to say unambiguously what the structures look like in the homogeneous direction(s) has led many to conclude that there is a fundamental problem with the orthogonal decomposition approach. While there may be, this is not it! The OD has identified for us a set of building blocks and is telling us that they are not always put together the same way. To see that this is not so uncommon in nature one need only consider the waves which occur on the open sea. The individual swell are made of many Fourier components, all propagating in different directions and at different rates. At the instant we observe them, they add up to form a structure which we identify as a typical wave. As we watch, the wave evolves and then disappears as the Fourier components which constitute it change. Does our inability to provide a precise recipe for the individual structure prevent us from learning about the dynamics of waves from studying the interaction of the building blocks? Of course not, and so most studies of the dynamics of the sea are based on Fourier analyis. In a similar manner, a study of the interaction of the building blocks yielded by an OD of a turbulent flow holds forth the prospect of teaching us much about the dynamics of turbulence.

We can gain considerable insight into the relation of the OD to coherent structures if we take an single eigenfunction and its coefficient from the OD, and transform it back into physical space. Choosing, for example, the lowest order term (n=0) from equation (16), the velocity field arising from it, say $u_i^{(0)}$, is given by

$$u_i^{(0)}(\underline{x},t) = \iint_{-\infty}^{\infty} a_0(k_1,\omega)\ \psi_i^{(0)}(x_2,x_3;k_1,\omega)\ e^{i(k_1x_1+\omega t)}dk_1d\omega \qquad (17)$$

Now the time and spatial extent in the x_1 direction for which $u_i^{(0)}$ appears coherent will be determined by the bandwidth of the disturbances constituting it. From equation (15),

$$<|a_n(k_1,\omega)|^2> = \lambda_n(k_1,\omega) \qquad (18)$$

Thus it is the eigenspectrum, λ_0, which will determine how long and over what distance the contribution of $u_i^{(0)}$ will appear coherent. If λ_0 is strongly peaked in wavenumber and frequency, then the contribution will appear highly coherent, persisting for a time proportional to the inverse bandwidth in frequency and over a distance proportional to the inverse bandwidth in wavenumber. If the contribution of this mode is spread smoothly over a broad band in either wavenumber or frequency, then its coherence will be proportionately decreased.

It is the length of time for which a coherent structure exists which determines its visibility, or our ability to see it. Similarly, it is the structures spatial extent which enables us to find it. Thus visibility and spatial extent are determined for stationary and partially homogeneous flows by the spectral content of the eigenspectra.

Let us assume for the moment that the OD does in some sense reflect what is happening in a real turbulent field, and that the eigenfunctions obtained really have captured the coherent features of the flow. Then the reason for the difference in visibility of coherent structures in high and low Reynolds number turbulence becomes readily apparent. At low Reynolds number, the energy (and hence the eigenspectra) is concentrated in a narrow band of frequency and wavenumber, usually not too far removed from those characterizing the instability and transition process. As a consequence, it should be (and is) relatively easy to see and find coherent structures in these flows. As the Reynolds number is increased, however, the eigenspectra are broadened by the non-linear interactions, and the coherent structures are harder to see and find. This does not mean that they are not present, only that they are evolving more rapidly.

RANDOM FIELDS WITH PERIODICITIES

A similar situation to that described above arises when the random field is periodic in one or more dimensions. Suppose for example that the flow is axially symmetric so that the single point statistical quantities are independent of the azimuthal coordinate, θ, and the two-point statistics depend only on differences in θ, say $\theta'-\theta$. Since the random field must be periodic in θ, it can be shown that the eigenfunctions satisfying equation (3) are harmonic functions of θ, i.e.,

$$e^{i\theta},\ e^{i2\theta},\ e^{i4\theta},\ \text{etc.}$$

Thus the orthogonal decomposition for axially symmetric processes reduces to the familiar Fourier series.

As for the homogeneous and stationary cases, it is most convenient to carry out the decomposition in the periodic direction first, then proceed with the remainder of the decomposition. Suppose, for example, the field is stationary in time, periodic in the azimuthal direction, θ, and inhomogeneous in the other two directions, say x and r. Then, the Fourier decomposition in the time variable yields the cross-spectrum, which must in turn be represented in terms of its azimuthal Fourier modes (Glauser 1987), i.e.,

$$A_{i,j}(x',x,r',r;\ \omega,\alpha) = \frac{1}{2\pi}\int_0^{2\pi} \Phi_{i,j}(x',x,r',r,\theta'-\theta;\omega)\, e^{-i\alpha(\theta'-\theta)}\, d(\theta'-\theta) \quad (19)$$

where $\Phi_{i,j}$ is the cross-spectrum and $A_{i,j}$ is its decomposition into azimuthal modes designated by the mode number $\alpha = 0,1,2,$ etc.

The orthogonal decomposition is thereby reduced to the following integral equation,

$$\iint_{\text{Domain}} \Phi_{i,j}(x',x,r',r;\omega,\alpha)\ \psi_j(x',r';\omega,\alpha)\ dx'dr' = \lambda(\omega,\alpha)\ \psi_i(x,r;\omega,\alpha) \tag{21}$$

The eigenspectra and eigenvectors are now functions of the frequency ω and the mode number α, as well as of the inhomogeneous variables x and r.

In a manner similar to the above, the Fourier decomposed velocity field, $\hat{u}_i$ can be represented in terms of the eigenfunctions by

$$\hat{u}_i(x,r;\omega,\alpha) = \sum_n a_n(\omega,\alpha)\ \psi_i^{(n)}(x,r;\omega,\alpha) \tag{22}$$

where the coefficients for each realization of the velocity field are computed from

$$a_n(\omega,\alpha) = \iint_{\text{Domain}} u_i(x,r,\theta,t)\ \psi_i^{(n)}(x,r;\omega,\alpha)\,dxdr \tag{23}$$

As before, the coefficients are related to the eigenspectra by

$$\langle a_n(\omega,\alpha)\,a_n^*(\omega,\alpha)\rangle = \lambda(\omega,\alpha)\ \delta_{mn} \tag{24}$$

The instantaneous and θ-dependent field can be recovered from equation (22) by performing the inverse transforms on $\hat{u}_i$, i.e.,

$$u_i(x,r,\theta,t) = \sum_n \left\{ \sum_{m=0}^{\infty} e^{i\alpha\theta} \left[\int_{-\infty}^{\infty} e^{i\omega t}\ a_n(\omega,\alpha)\,\psi_i^{(n)}(x,r;\omega,\alpha)\,d\omega \right] \right\} \tag{25}$$

Immediately, we see that we have the same problem we encountered above where the individual eigenfunctions arising from the decomposition in the inhomgeneous directions to not contribute in a simple way to the velocity field, since each eigenfunction is multiplied by a random coefficient which is a function of the frequency and mode number. Thus, just as for the homogeneous and stationary case, it is not possible, without additional assumptions to identify a single coherent structure, since its composition will continually evolve. The situation in the θ-direction is a little better than before, at least conceptually, since it is easier to visualize the roles of individual azimuthal modes than it is continuously varying ones. The zeroth mode is simply the same for all θ, while the higher order modes have a sinusoidal variation in θ, the number of oscillations corresponding to the mode number.

It is easy to see that the presence of a coefficient which is dependent on both mode number and frequency allows an individual coherent structure to be transformed from one modal shape to another during its life-time. If we believe that coherent structures are vortical structures which evolve in time and space, the OD the would seem to have retained a feature essential for a successful representation. This situation can be contrasted with one where the coherent structure simply stays in one place and decays, which could conceivably occur only when the flow is completely inhomogeneous and non-stationary.

Given that the effect of periodicities, homogenieties, and stationarity is to make the eigenfunctions separable into harmonic and non-harmonic parts, it is difficult to see any other way for the OD to be able to describe such an evolution than by the result we have obtained. This point is to be emphasized here, because some have suggested that these random coefficients represent a deficiency of the decomposition, and have discouraged its use. Certainly, the problem is more complex than we might have wished, but it does appear that the kinematics of an evolving flow are accurately described, and probably in as simple manner as possible.

PARTIAL DECOMPOSITIONS AND EXPERIMENTS

In spite of the fact that the OD and most of our understanding of it have been around now for 22 years or so, there has been no real opportunity to find out what it tells us about coherent structures in turbulence. This is because acquiring the information on the cross-correlation or cross-spectral tensor throughout the flow field has proven beyond the vision of most experimenters. The situation has, however, improved remarkably over the past few years. First, two experimental programs have been completed which contain enough information to carry out the decomposition in three out of the four space-time dimensions necessary for a complete description (Herzog 1986, Glauser 1987). These are reviewed in below. Second, the advent of full Navier-Stokes computer simulations of simple flows has opened up new possibilities which are beginning to be exploited (e.g., Moin 1984). Unfortunately, a considerable effort has been spent (or mis-spent) trying to figure out what the eddies look like, instead of using the information we already have to study the far more important dynamical characteristics of coherent structures. Happily, this is also beginning to change (e.g.,Aubry 1987).

Before reviewing the results of the experimental program, it is worth reviewing precisely what information is desirable to fully exploit the OD, and what one gives up by only having some of it. As is obvious from equations (3), (11), and (21), sufficient information on the cross-correlation or cross-spectral tensor is necessary to allow the eigenvalue problem to be solved. This will yield the eigenvalues and eigenvectors. Perhaps not so obvious is that then, a complete space-time realization of the velocity field is necessary to allow the coefficients of equations (7), (14), or (23) to be computed. Only when this latter element is included is there sufficient information to follow the space-time evolution of the individual structures. It doesn't require much knowledge of experimental capabilities to see why the application of the OD has been rather slow in developing. It should also be easy to see why the continuing development of numerical simulation capabilities is of great interest in this context. The same can be said for the development of holographic anemometry techniques which allow an entire field to be measured simultaneously.

Even in the absence of such complete information, however, it has been possible to glean a great deal of information about the nature of these two different turbulent flows by applying the OD to experiments which provided only part of the needed information. These experiments will be discussed in the following sections. The intent here is not to present the results in detail, which has already been done elsewhere, but to focus on what even a partial decomposition is able to say about the character of structures in the flow. We begin with the experiment of Glauser in a high Reynolds number axisymmetric jet mixing layer, then consider the experiment of Herzog in a low Reynolds number pipe flow.

THE AXISYMMETRIC JET MIXING LAYER

The experiment of Glauser (1987) was carried out in the mixing layer of an axisymmetric air jet. The Reynolds number based on jet exit velocity and diameter was 10^5, and the Mach number was 0.067. The exit velocity profile was flat and the boundary layer at the lip of the jet was laminar. The experiment was performed using rakes of up to 14 hot-wire probes so that the flow could be simultaneously measured at many locations. The experiment was performed in phases of increasing complexity so that the results of simple decompostions could be studied before carrying out additional measurements. All of the two-point cross-spectra involving the streamwise and radial velocity components were measured on a grid with seven positions across the layer, and 25 azimuthal positions from 0 - 180°. Thus, in all 5000 cross-spectra were measured so that with the frequency dependence 10 x 10^6 cross-spectral entries were used in the eigenvalue analysis.

The first partial decomposition considered only the radial and time variation of the velocity field. Therefore, since the field was stationary in time, the cross-spectrum as a function of radius and frequency only was used to carry out the eigenvalue problem, i.e.,

$$\int_{R_1}^{R_2} \Phi_{1,1}(r',r;\omega)\,\psi_1^{(n)}(r;\omega)\,dr = \lambda^{(n)}(\omega)\ \psi_1^{(n)}(r;\omega) \tag{26}$$

where R_1 and R_2 were chosen to span the mixing layer. Note that the dependence on x',x, and θ',θ has been suppressed since these were all fixed. Thus the results are equivalent to summing over all the azimuthal modes for a single domnstream position.

Figure (1) shows the eigenspectra for the first three modes, and shows clearly the dominant nature of the first which integrates to 40% of the energy in the field. From even the partial decomposition it is possible to reconstruct the velocity spectrum using

$$\Phi_{1,1}(r;\omega) = \sum_n \lambda_n \left|\psi_1^{(n)}(r;\omega)\right|^2 \tag{27}$$

Figure (2) shows such a reconstruction of the one-dimensional spectrum for a position near the center of the mixing layer, and it is apparent that only three terms are required to completely recover it. The individual terms of the spectrum at each position can be integrated with respect to frequency to show what fraction of the variance at each position across the layer is contributed by each term. Figure (3) shows the results of such an integration for the first term.

An even more striking illustration of the ability of the OD to capture the details of the flow is given by Figure (4) which shows how a single realization of the real part of the Fourier transform of the instantaneous signal is made up by the individual terms. Even the first term alone has captured most of the salient features, while three terms recover virtually all. In view of this, there really should be no more talk of the inability of the OD retain phase information, or of it smearing out significant events.

A reconstruction of the azimuthal variation of the field, together with the time information, would answer many question about the behavior of coherent structures in the jet mixing layer. Unfortunately, because it was not possible to acquire data at all values of θ and r simultaneously, it was not possible to evaluate the coefficients as in equation (23). Even so, the OD applied to the azimuthal direction provides some tantalizing clues. Figures (5) - (7) show the modal constituents of two components of the velocity correlation tensor, $B_{1,1}$ and $B_{1,2}$. On the high speed (or inside of the mixing layer), both normal and Reynolds stresses show almost a complete dependence on the zeroth order mode (Figures 5a and 5b), while on the low speed side (or outside) the dependence has shifted almost entirely to a band of modes centered around m = 5 or 6 (Figures 7a and 7b). In the center (Figures 6a and 6b), the normal stress component, $B_{1,1}$ hascontributions from both m = 0 and the band at m = 5 - 6, but the Reynolds stress, $B_{1,2}$ has nearly uniform dependence on everymode <u>but</u> m = 0. The first three eigenspectra for modes 1,2,3 and 5 are shown in Figure (8 a-d). Immediately apparent is the fact that the first eigenspectrum for each mode is dominate, the same being true for modes 4, 6, 7, and 8 which are not shown here. Also clear is the shift toward lower frequencies with increasing mode number.

One of the problems with a partial decomposition which does not include the streamwise variation is that if time is included, it has contributions from the streamwise variation because the disturbances are convected by the mean stream. Thus the shift of the higher order eigenspectra to lower frequencies probably is consistent with the dominance of the higher order modes on the low speed side of mixing layer. Glauser and George (1987b) were able to use these obvervations to propose a life-cycle for the generation and decay of the coherent stuctures in the axisymmetric jet mixing layer. In particular, they suggested that:

> "Vortex ring-like concentrations arise from an instability of the base flow, the induced velocities from vortices which have already formed

providing the perturbation for those which follow. These pairs of vortices then behave like the text book examples of inviscid rings, the rear vortex ring overtaking the vortex ring ahead of it, the rearward vortex being reduced in radius and the forward one being expanded by their mutual interaction. The rearward ring is stabilized by the reduction in its vorticity (by compression) thus the predominance of the 0th mode on the high speed side (core region). The forward ring has its vorticity increased by stretching as it expands in radius. This narrowing of its core while the radius is expanding causes the vortex to become unstable, thus the predominance of the 4th-6th modes from the center of the shear layer outwards. The continued effect of the rearward vortex on the forward one and the interaction of the now highly distorted ring with itself, accelerates the instability until its vorticity is now entirely in small scale motions, in effect an energy cascade from modes 4-6 all the way to the dissipative scales. This incoherent turbulence is swept from the outside where it has been carried, back to the center of the mixing layer as the still intact rearward vortex passes. It is this collecting of the debris, both small scale vorticity and fluid material, which has been recognized as 'pairing'. The entire process is repeated as a new rearward vortex overtakes and destabilizes the one ahead of it."

In order to evaluate whether or not this picture is correct will require application of more complete versions of the OD. The most useful and straight-forward experiment would yield information on enough points in θ and r simultaneously to enable the $a_n(\omega,\alpha)$ to be computed. Then the instantaneous field could be reconstructed in order to see the sequence by which the various modes contribute in time. Certainly there is room here also for a refined application of the shot-noise decomposition, the discussion of which has been avoided in order to avoid confusing it with the OD. Finally, and perhaps most interestingly, the decomposition should be applied to the streamwise direction so that the space-time evolution can be studied together without the problem introduced by the mean convection.

THE VISCOUS SUBLAYER

The second and last experiment to be considered here is that carried out by Herzog (1986) in the wall region of low Reynolds number turbulent pipe flow. The experiment was an extension of the earlier one of Bakewell (1966). The experiment was realized in a 12" pipe flow using glycerine as the working fluid. The Reynolds number based on pipe diameter and bulk flow velocity was 8750, and the flow was tripped at the pipe entrance to insure fully developed flow at the measurement locations. All of the measurements were taken below $y^+ = 40$. While the experimental techniques were quite different than those used by Glauser (1987), the overall complexity of the experiment was of comparable magnitude.

In Bakewell's original experiment only the cross-correlations involving the streamwise velocity component were measured as functions of the time, radial and spanwise coordinates. Herzog measured all components of the cross-correlations, and included the streamwise dependence, but not the time. Therefore all correlations in the latter experiment correspond to zero time lag. Thus the measured correlations were functions of two homogeneous coordinates, r_1 and r_3, representing the streamwise and spanwise separations, and the inhomogeneous coordinates, x_2'and x_2. (Note that it was decided to treat the spanwise direction as homogeneous instead of utilizing the azimuthal symmetry of the flow because of the small thickness of the sublayer relative to the pipe circumference.)

Figure (9) shows lowest order eigenvector (denoted by Herzog as $\lambda^{(1)}$) while Figures (10 a and b) show the cross-sections through the k_1 and k_3 axes of the first three eigenspectra. As with Bakewell's and Glauser's experiments, the first eigenspectrum is dominant at all but the very highest wavenumbers. Figures (11a through d) show how the variances of the three velocity components and the Reynolds stress are distributed between the first three terms of the OD. Particularly of interest is that the first term accounts for almost all of the streamwise velocity variance and the Reynolds stress near $x_2^+ = 20$.

The off-axis peak in the lowest order eigenspectrum (Figure 9) at $k_3^+ = 0.0035$ corresponds to a span-wise wavelength in wall units of $\lambda^+ = 286$. (Note that Herzog defines $k=1/\lambda$ omitting the usual 2π.) Herzog notes that the distance between the nodes associated with this wavelength would be half of this value which is very close to the oft-reported wall streak spacing. It is also clear from the bandwidth associated with the streamwise wavenumber variation that the streamwise extent of the coherent structure is very large indeed, perhaps as much as an order of magnitude greater than its spanwise extent. It should be noted that these observations are not dependent on an ad hoc application of the shot-noise decomposition, but depend only on the original hypothesis which led to the OD itself.

Note that it can not be inferred directly from the results of the OD that there are counter-rotating rolls of the type inferred originally by Bakewell (1966) and subsequently using somewhat different arguments by Herzog (1986). These inferences depended entirely on the introduction of other assumptions about the field and on the use of the shot-noise decomposition. Whether or not such rolls are really present depends very much on the nature of the coefficients of the reconstructed field, which in this case will depend on k_1, k_3, and also ω since the field is stationary. If, in fact, the eigenfunctions add up in constant phase as these authors suggest, then there will be counter-rotating rolls. On the other hand, when the full time-dependence of the stationary field is included with effect of the streamwise and spanwise homogeneities, the longitudinally extended and highly coherent structure may appear quite differently. It might appear so different in fact that it is scarely recognized as such in a real flow.

It should be clear that, as for the mixing layer, the part of the puzzle as to what the OD is really telling us can be filled in only by determining the coefficients for a realization of the flow field. Only by doing so can we see visually how the building blocks are being added by the flow itself. This determination, as for the mixing layer above, requires information from many points simultaneously. Certainly this would seem well within the scope of modern numerical simualtions.

SUMMARY AND CONCLUSIONS

The elements of the application of proper orthogonal decomposition techniques have been reviewed. Particular attention has been placed on what can be gained from the decomposition, and what it can tell us about the nature of coherent structures in turbulence. Two experiments have been used to illustrate the theory, one by Glauser (1987) in the axisymmetric jet mixing layer,and the other by Herzog (1986) in the viscous sublayer of a turbulent pipe flow. Both experiments show clear evidence of the existence of coherent structures, and give a strong indication that they play an important role in the dynamics of the turbulence.

There are, of course, many ways to detect the presence of coherent structures. If detection were the only contribution of the orthogonal decomposition, it would not be worth the effort. What the orthogonal decomposition provides is a mathematical description which can be used for further analysis. The results of a decomposition can be used in the governing equations to understand the interactions of the flow with itself. Or they can be used to study the effect of coherent structures on other phenomena of interest, like noise or mass transport (v. Arndt and George 1974). The eigenfunctions can be used as the basis for other dynamical calculations in order to capture the essence of the turbulence in a simplified (and often more instructive) analysis (v. Aubry 1987). Finally, the orthogonal decomposition can be applied directly to the Navier-Stokes equations to develop a hierarchy of equations which can be solved directly with appropriate closure models for the higher order terms. This latter may be the most important of all, for it holds forth the hope of engineering calculations which include in a direct manner the dynamical influences of the coherent structures.

ACKNOWLEDGEMENTS

The author is particularly grateful to Professor Mark Glauser (of Clarkson University) with whom he has collaborated for a number of years. He would also like to express his appreciation to Dr. Siegfried Herzog (of the Pennsylvania State University) for permission to use the figures from his work, and to Dr. Hans Saustrup Kristensen (of the Danish Technical University) for calling his attention to the cartoon by Piet Hein. Finally, support of the Danish - American Fulbright Commission, the Danish Technical University, the University at Buffalo, SUNY, and the Calspan UB Research Center during this sabbatical period is gratefully acknowledged.

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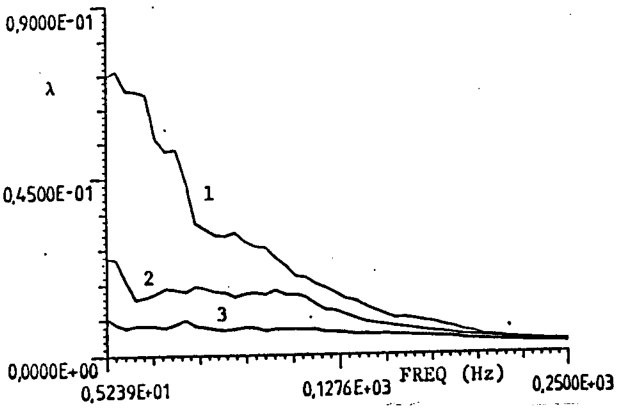

Fig. 1 Eigenspectra for Jet Mixing Layer, x/D = 3 (Glauser 1987).

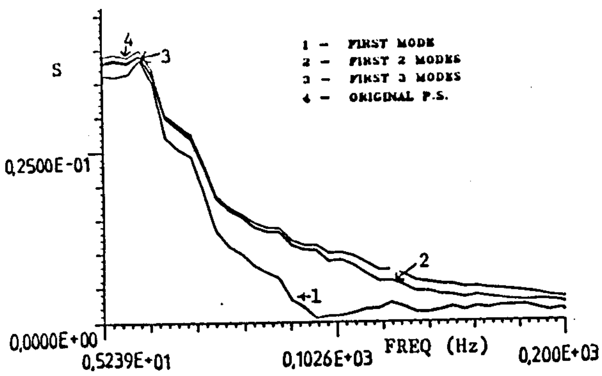

Fig. 2 Reconstruction of Power Spectrum, r/R = 0.46.

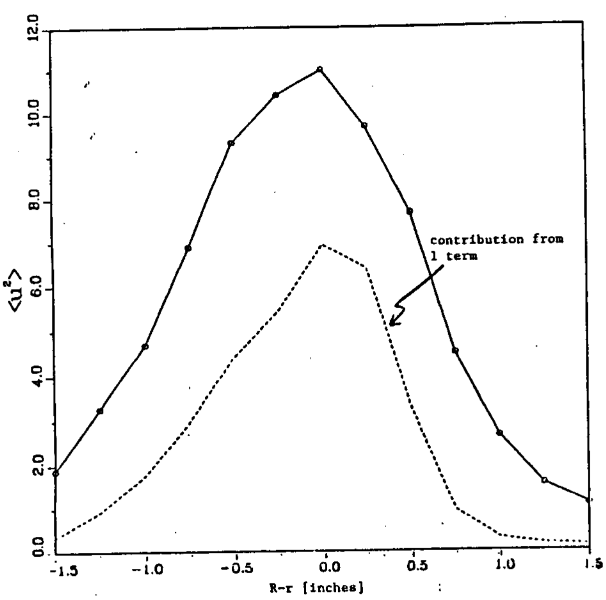

Fig. 3 Total $\langle u^2 \rangle$ and Contribution From First Term in OD.

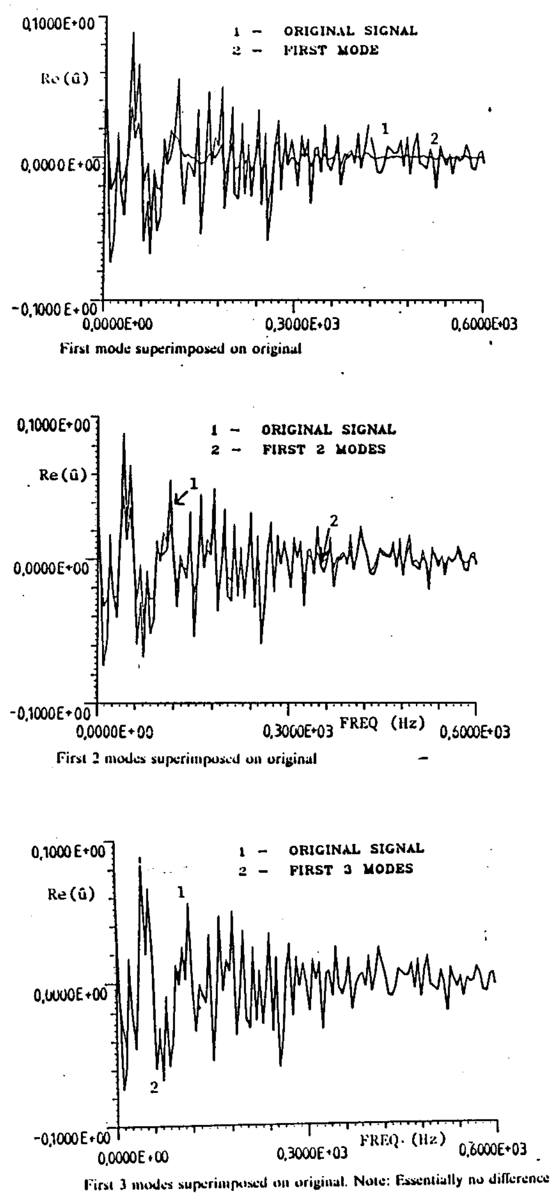

Fig. 4 Reconstruction of $\mathrm{Re}\{\hat{u}_1(\omega)\}$.

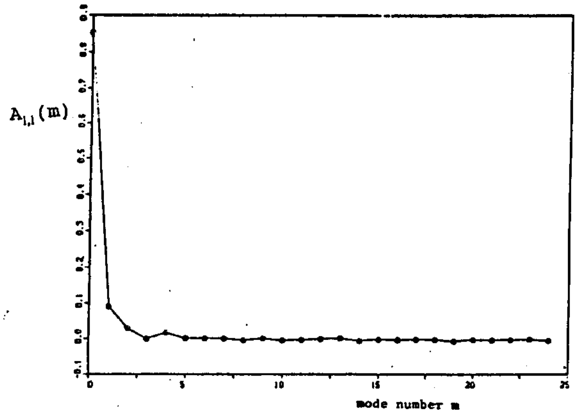

Fig. 5a $A_{1,1}$ at r/D = 0.13.

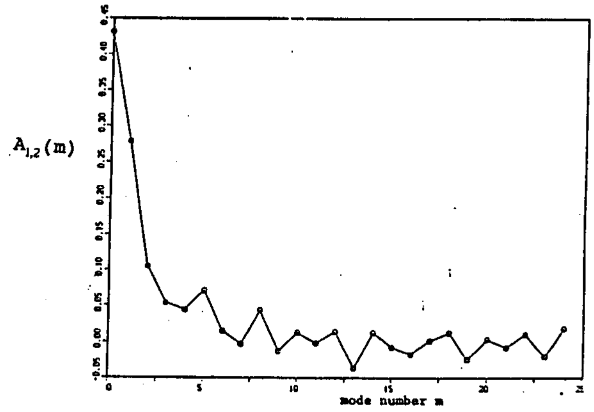

Fig. 5b $A_{1,2}$ at r/D = 0.13.

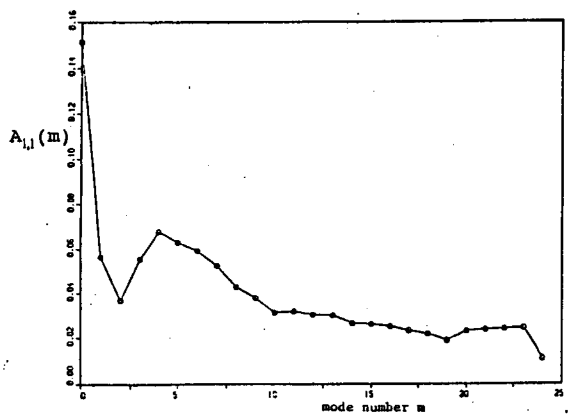

Fig. 6a $A_{1,1}$ at r/D = 0.35.

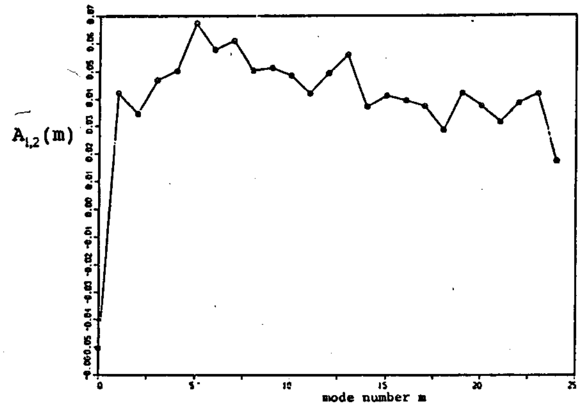

Fig. 6b $A_{1,2}$ at r/D = 0.35.

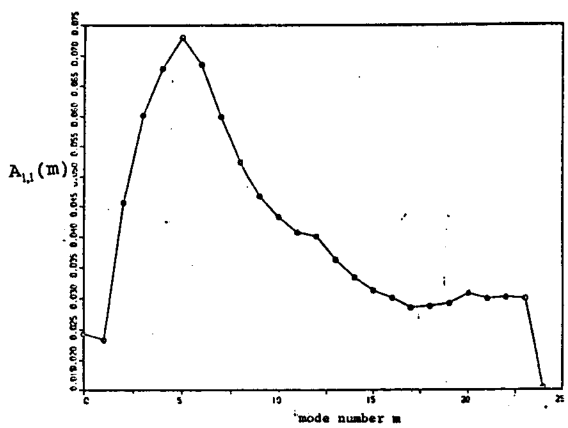

Fig. 7a $A_{1,1}$ at r/D = 0.46.

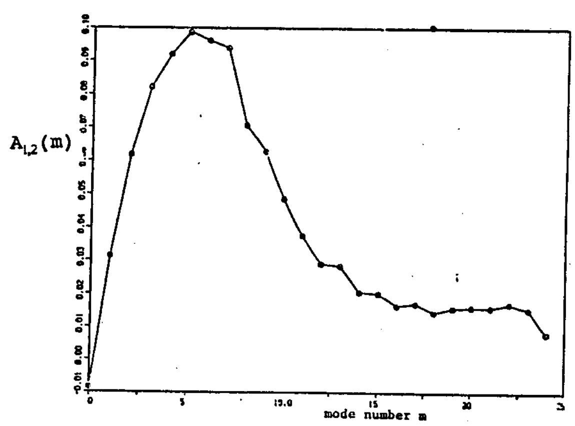

Fig. 7b $A_{1,2}$ at r/D = 0.57.

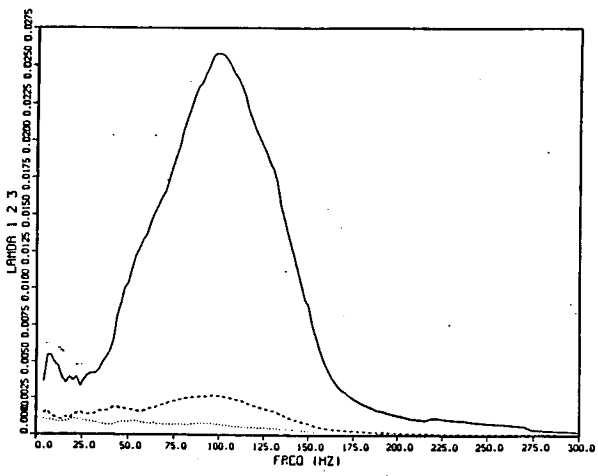

Fig. 8a First 3 Eigenspectra, Mode 0.

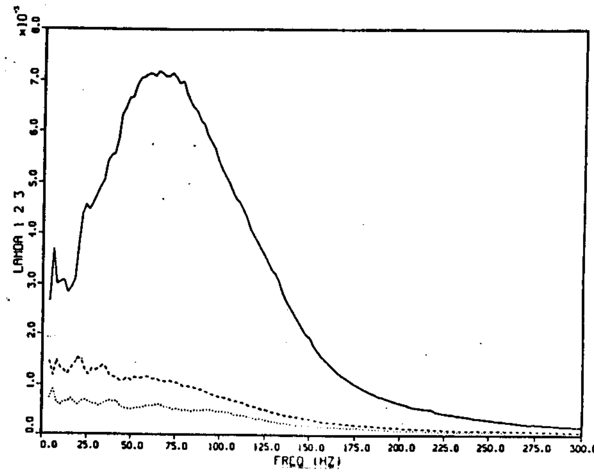

Fig. 8b First 3 Eigenspectra, Mode 1.

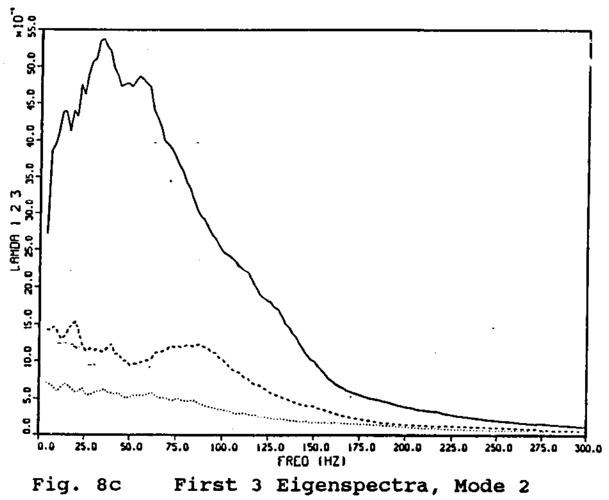

Fig. 8c First 3 Eigenspectra, Mode 2

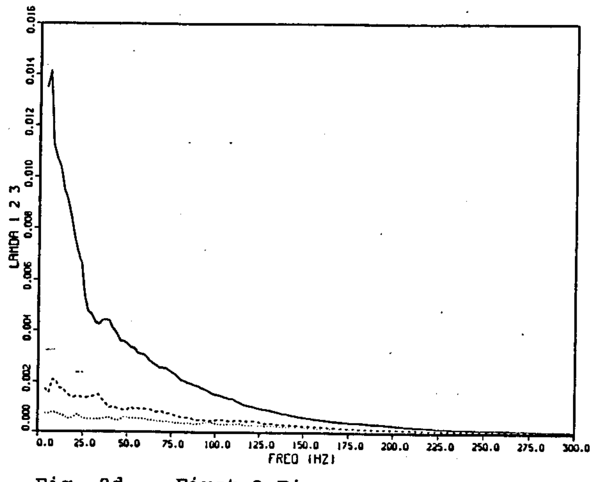

Fig. 8d First 3 Eigenspectra, Mode 5.

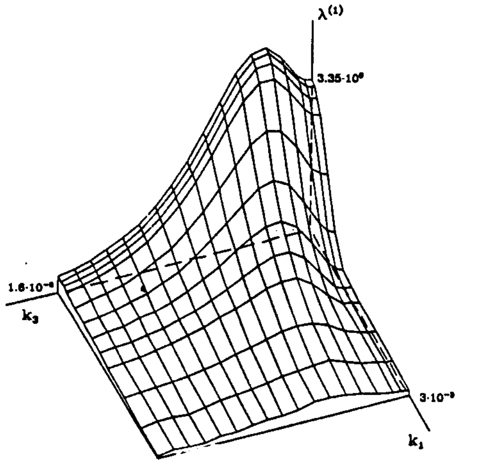

Fig. 9 Dominant Eigenspectrum for Wall Layer (Herzog 1986).

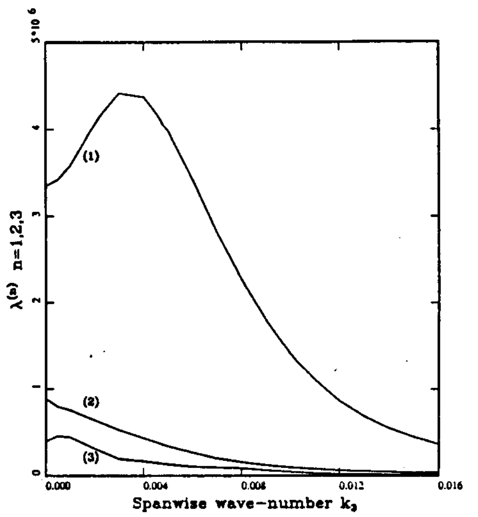

Fig. 10a First 3 Eigenspectra as Function of spanwise wavenumber, k_3.

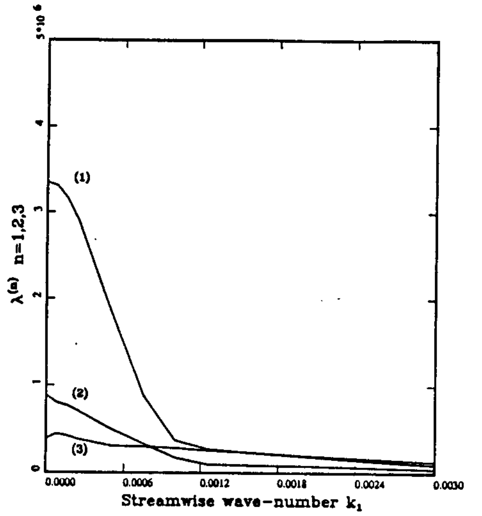

Fig. 10b First 3 Eigenspectra as Function of streamwise wavenumber, k_1.

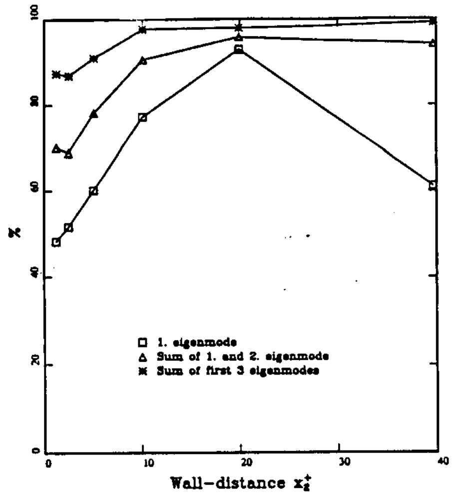

Fig. 11a Contributions of OD terms to <u^2>.

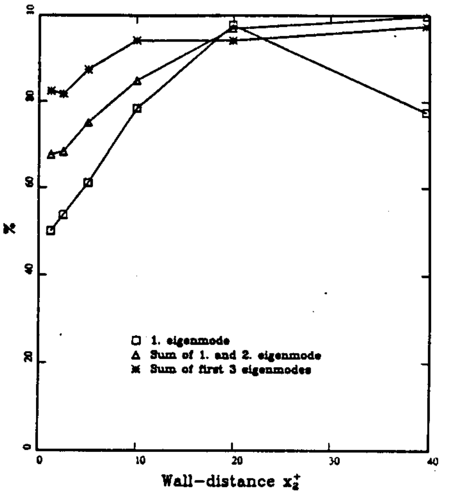

Fig. 11b Contributions of OD terms to <uv>.

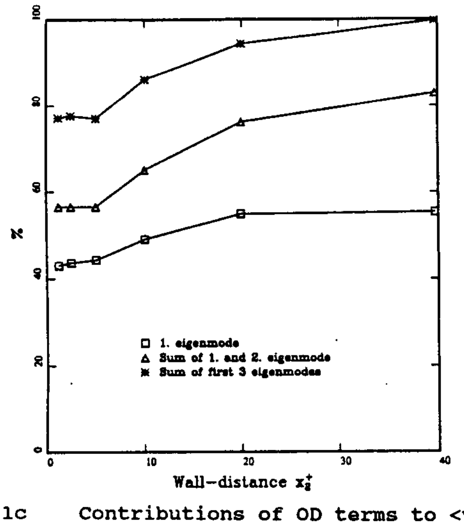

Fig. 11c Contributions of OD terms to <v^2>.

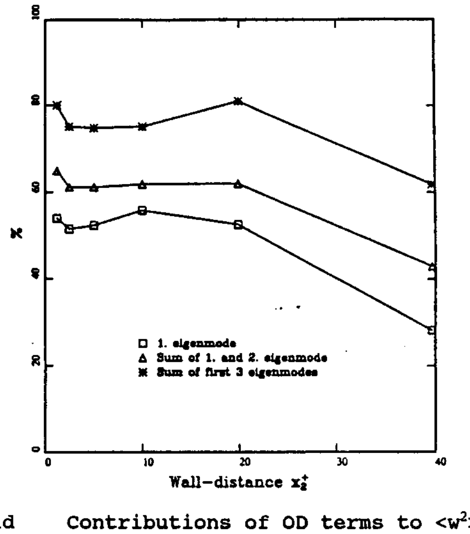

Fig. 11d Contributions of OD terms to <w^2>.