The Self-Preservation of Turbulent Flows and Its Relation to Initial Conditions and Coherent Structures

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INTRODUCTION

It is difficult to trace the origins of the ideas of self-preservation and similarity solutions, and their application to fluid mechanics. They appear to have first been applied to turbulence by Zel'dovich (1937), but had been used long before in laminar boundary layer theory by Blasius (1908). Self-preservation (or similarity) is said to occur when the profiles of velocity (or any other quantity) can be brought into congruence by simple scale factors which depend on only one of the variables. A consequence of self-preservation is that the dynamical equations become independent of that variable and are thereby reduced by one variable in their functional dependence. A major benefit occurs if the original equations are two-dimensional or axisymmetrical since self-preservation reduces the governing equations to ordinary differential equations.

More physical insight into the dynamical significance of self-preservation can be had if an alternative definition is used. A flow is said to be self-preserving if there exist solutions to its dynamical equations and boundary conditions for which, throughout its evolution, all terms (of dynamical significance) have the same relative value at the same relative location. Thus self-preservation implies that the flow has reached a kind of equilibrium where all of its dynamical influences evolve together, and no further relative dynamical readjustment is necessary. It should be clear that no flow can be expected to begin in a self-preserving state (unless particular care were taken in its initiation) since some dynamical readjustment is always necessary to smooth out the details of the initial conditions (infinite or very large gradients, for example). Self-preservation is therefore an asymptotic state which a particular flow attains once its internal readjustments are complete.

There has been widespread belief in the turbulence community (v. Townsend 1976) that flows achieve a self-preserving state by becoming asymptotically independent of their initial conditions. Thus, for example, all jets should asymptotically grow at the same rates, all wakes should be independent of their generators, and so forth. Such an argument is a logical consequence of a belief that 'turbulence forgets its origins', and can be modeled by its local properties. It is this belief which forms the basis of all single point models for turbulence.

Unfortunately, over the past two decades there has been increasing experimental evidence that such a simple picture is not correct, and that, in fact, even flows which appear to scale in similarity variables (like centerline velocity and half-width) are dependent on their initial conditions. For example, Wygnanski et al. 1986, show have very different growth rates for the wakes behind cylinders and screens, even though both appear to be self-preserving. Similarly, a variety of growth rates and profiles have been reported for jets, plumes, mixing layers and most other free turbulent shear flows by different experimenters. While some dismiss these results as being inconclusive or due to improper experimental techniques, others use them to argue that the ideas of self-preservation are not relevant to real turbulent flows.

If the ideas of self-preservation were to be abandoned for free shear flows because of conflicting experimental evidence, it would not be the first time for turbulence. Such was exactly the case for the von Karman-Howarth (1938) analysis of decaying turbulence, and its failure to describe the actual decay observed for the turbulence behind grids in wind tunnels. It was this failure, and Batchelor's subsequent reanalysis (Batchelor 1948), which argued that turbulence could never be scaled by a single length scale, and led to our contemporary view of turbulence at high Reynolds numbers as a multi-length scale phenomenon requiring at least separate scales for the energy-containing eddies and for the dissipative scales. Kolmogorov's ideas for the local self-preservation of the smallest scales and the concept of the universal equilibrium range in part gained their acceptance because of the failure of the more general self-preservation arguments.*

THE PURPOSE OF THIS PAPER

The fundamental premise of this paper is that the concept of full self-preservation <u>does</u> represent an important concept in turbulence theory. It will be argued that the problems presented by the experiments lie not in the experiments themselves nor in the concept of self-preservation, but rather in the restrictive manner in which the self-preservation analyses have been carried out. It will be shown that a more general self-preservation analysis leads to the conclusion that, contrary to previous belief, there exists a multiplicity of self-preserving states (for a particular type of flow) and that each state is uniquely determined by its initial conditions.

The more general type of analysis will be carried out in detail for the axisymmetric jet and the plane wake, and shown to be consistent with the experimental observations. Application of the technique to the axisymmetric wake will be seen to lead to some interesting new possibilities, also anticipated by experiments. In addition, the results of another look at the self-preservation analyses of homogeneous flows will be cited which raise serious questions about our understanding of small scale turbulence.

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*As will be noted later, these analyses and the conclusions drawn from them have recently been challenged.

A consequence of these new analyses will be the recognition that several kinds of self-preservation are possible. In particular:

(i) Flows can be fully self-preserving at all orders of the turbulence moments and at all scales of motion.

(ii) Flows can be partially self-preserving in that they are self-preserving at the level of the mean momentum equations only, or up to only certain orders of the turbulence moments or at certain scales.

(iii) Flows can be only locally self-preserving in the sense that the profiles appear to scale with local quantities, but the equations of motion do not admit to self-preserving solutions.

The recognition of these various levels of self-preservation which are possible leads to the following conjectures as to which will, in fact, asymptotically describe the flow:

Conjecture I: If the equations of motion, boundary and initial conditions admit to self-preserving solutions, then the flow will always asymptotically behave in this manner.

Conjecture II: If the equations, boundary and initial conditions governing the flow do not admit to fully self-preserving solutions, the flow will adjust itself as closely as possible to a state of full self-preservation.

While neither of these conjectures are proven, they are believed to be consistent with the body of turbulence data. Conjecture II includes among its possibilities partial self-preservation (as will be demonstrated later) and local self-preservation.

It will further be argued that there is a link between self-preservation and coherent structures. In fact, it will be suggested that it is this link which may provide the clue to the proof of the conjectures above and to the turbulence problem in general.

THE TURBULENT JET

The mean flow of an axisymmetric jet issuing into a quiescent environment (Figure 1) is described to first order by the equations

$$U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = - \frac{1}{r} \frac{\partial}{\partial r} r\overline{uv} \tag{1}$$

and

$$\frac{\partial U}{\partial x} + \frac{1}{r} \frac{\partial rV}{\partial r} = 0 \tag{2}$$

The second equation can be solved to yield the radial velocity as

$$V = -\frac{1}{r}\int_0^r \left[\frac{\partial U}{\partial x} \right] dr' \tag{3}$$

This can in turn be used to eliminate V from equation (1).

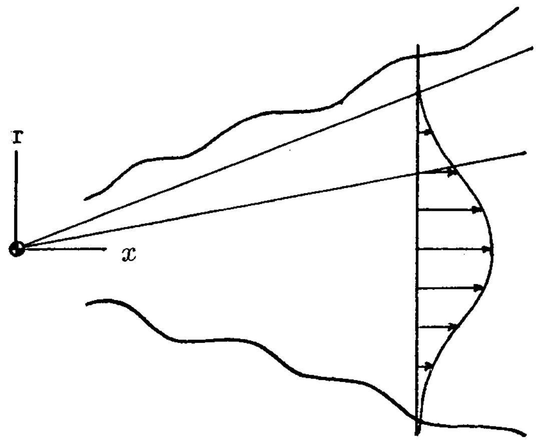

Figure 1. Sketch showing development of jet.

The momentum equation (1) can be integrated across the flow to yield

$$\frac{d}{dx}\int_{-\infty}^{\infty} \left[U^2 + \overline{u^2} - \frac{\overline{v^2}+\overline{w^2}}{2} \right] rdr = 0 \tag{4}$$

The second order terms have been retained here since they can be important in evaluating experimental data. It follows from integration that the momentum integral must be constant and equal to the rate at which momentum is added at the source, say ρM_o where ρ is the density and M_o is the kinematic momentum rate. Thus,

$$M_o = 2\pi\int_0^{\infty} \left[U^2 + \overline{u^2} - \frac{\overline{v^2}+\overline{w^2}}{2} \right] rdr \tag{5}$$

We seek self-preserving solutions of the form

$$U = U_s(x) \quad f(\eta) \tag{6}$$

and

$$-\overline{uv} = R_s(x) \; g_{12}(\eta) \tag{7}$$

where

$$\eta = r/\delta \tag{8}$$

and

$$\delta = \delta(x) \tag{9}$$

The profile functions $f(\eta)$ and $g_{12}(\eta)$ account for all of the radial variation, and their existence implies that the profiles of velocity and Reynolds stress at every downstream location can be collapsed into single curves. All of the streamwise variation has been incorporated into the functions $U_s(x)$, $R_s(x)$ and $\delta(x)$ which must be determined so that *all of the terms in the dynamical equation maintain the same relative balance at the same relative location.* If such solutions are possible, an equilibrium between terms has been established and the flow evolves in a highly structured manner so that the flow can be said to be self-preserving.

THE TRADITIONAL APPROACH

It is at this point in the analysis that the crucial assumptions are usually made which dictate the classical solutions for the self-preserving jet. For example, Monin and Yaglom (1972) argue that for a jet issuing from a point source of momentum, the scale quantities U_s, R_s and δ must be entirely determined by the rate at which momentum is added at the source, ρM_o, and the distance downstream, x. That is,

$$U_s = U_s(x, \rho M_o) \tag{10}$$

$$R_s = R_s(x, \rho M_o) \tag{11}$$

and

$$\delta = \delta(x, \rho M_o) \tag{12}$$

On dimensional grounds it follows immediately that

$$\delta \sim x \tag{13}$$

$$U_s \sim M_o^{1/2}/x \tag{14}$$

and

$$R_s \sim M_o/x^2 \sim U_s^2 \tag{15}$$

It is clear at even this early point in the analysis that if the equations admit to self-preserving solutions at all, they must spread linearly and the centerline velocity must decay as x^{-1}. Moreover, only a single solution is possible as there are no other parameters, thus the idea of a 'universal' asymptotic behavior for jets which is independent of the details of the initial conditions.

Another approach which has commonly been used (v. Townsend 1976, Tennekes and Lumley 1972) does not begin with dimensional analysis, but equally predetermines the final form of the solution by arguing that the solution depends only on a single length scale and single velocity scale so that $R_s = U_s^2$. Interestingly enough, Hinze (1975) avoids both these lines of reasoning, but then forces the solution to the same final form by invoking an eddy viscosity assumption.

Carrying out the appropriate differentiations of equations (6) through (9) and substituting into the governing equations (1) through (3) yields the transformed equations as,

$$\left[U_s \frac{dU_s}{dx}\right] f^2 - \left\{\left[U_s \frac{dU_s}{dx}\right] + 2\left[\frac{U_s^2}{\delta}\frac{d\delta}{dx}\right]\right\} \frac{f'}{\eta} \int_0^\eta f\, \bar{\eta} d\bar{\eta}$$

$$= \left[\frac{R_s}{\delta}\right] \frac{(\eta g_{12})'}{\eta} \tag{16}$$

This can be multiplied by δ/U_s^2 to obtain,

$$\left[\frac{\delta}{U}\frac{dU_s}{dx}\right] f^2 - \left\{\left[\frac{\delta}{U}\frac{dU_s}{dx}\right] + 2\left[\frac{d\delta}{dx}\right]\right\} \frac{f'}{\eta} \int_0^\eta f\, \bar{\eta} d\bar{\eta}$$

$$= \left[\frac{R_s}{U_s^2}\right] \frac{(\eta g_{12})'}{\eta} \tag{17}$$

Note that for a self-preserving solution to exist, all of the terms in brackets must have the *same* x-dependence.

The momentum integral, equation (5) can be similarly transformed to yield

$$M_o = \left[U_s^2 \delta^2\right] 2\pi \int_0^\infty f^2(\eta)\eta d\eta \tag{18}$$

where the second order terms have been neglected. (Note that this neglect is not essential to the subsequent arguments and they could have easily been retained by scaling them with R_s.)

The restrictive nature of the assumptions described above now becomes clear, for if either $\delta \sim x$ or $R_s \sim U_s^2$, all of the terms in brackets must be *constant* since $d\delta/dx \sim$ constant or $R_s/U_s^2 \sim$ constant respectively. For the latter, the remaining constraints for self-preservation are,

$$\frac{\delta}{U_s}\frac{dU_s}{dx} \sim \text{constant} \tag{19}$$

and

$$\frac{d\delta}{dx} \sim \text{constant} \tag{20}$$

Equation (20) requires $\delta \sim x$ while equation (19) admits to solutions of the form $U_s \sim x^m$ where m is at this point arbitrary. The momentum integral constraint, however, requires

$$\left[U_s^2\ \delta^2\right] \sim \text{constant} \tag{21}$$

from which follows that m=-1 is the only possible self-preserving solution.

Thus both traditional lines of reasoning have yielded the following self- preserving forms:

$$U = \frac{M_o^{1/2}}{x}\ f(\eta) \tag{22}$$

$$-\overline{uv} = \frac{M_o}{x^2}\ g_{12}(\eta) \tag{23}$$

$$\delta = x \tag{24}$$

where the constant of proportionality in equation (24) is chosen as unity, the scale factor being absorbed into g_{12}. The governing equation reduces to

$$-\ f^2 - \frac{f'}{\eta}\int_0^{\eta} f\ \bar{\eta}d\bar{\eta} = \frac{(\eta g_{12})'}{\eta} \tag{25}$$

Thus there is only one solution which represents the jet from a point source of momentum.

Real Jets

Jets which occur in nature are not the result of pure sources of momentum, but rather emanate from sources of finite dimensions providing also mass and energy to the flow at a finite rate. Measurements in jets generated by such sources, however, often appear to asymptotically exhibit self-preserving behavior. It has been argued that at large distances from the source, all jets should become asymptotically independent of the details of the source and depend only on the rate at which momentum is added and the distance from the source. Thus all jets have been expected to behave as jets arising from a point source of momentum only (v. Fischer et al. 1979). If all jets become asymptotically independent of the initial conditions, then it is obvious that they must all decay at the *same rate* and exhibit the *same shape* (once an adjustment for origin has been made), and these must correspond to the equivalent point source of momentum.

A rationale for the above can best be illustrated by considering a jet which has both a finite source of mass, say ρm_o, and momentum, ρM_o. The rate of flow of mass at any cross-section of the flow is illustrated in Figure (2). As the distance from the source is increased, so does the net rate of mass flow at any cross-section due to entrainment. Thus at large distances from the source the mass flow arising from entrainment swamps the initial mass flow. This is quite unlike the momentum (or momentum integral) which is constant. Thus it can be argued that asymptotically the initial rate of mass addition is unimportant. Similar lines of reasoning can be invoked for other kinds of source conditions, including various kinds of forcing, since it can be argued that the turbulence should "forget" its origins as it evolves, and the starting conditions should be overwhelmed by subsequent developments.

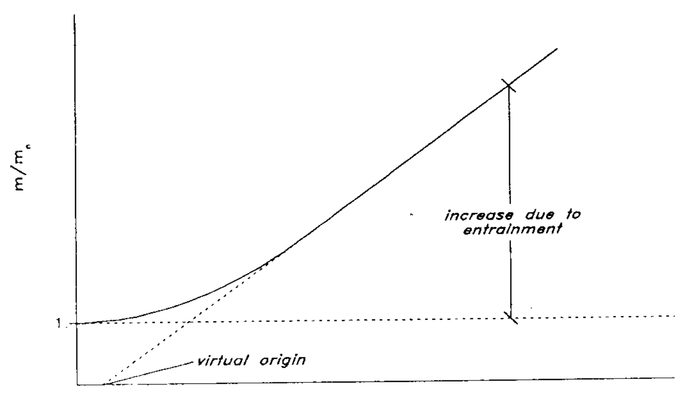

Figure 2. Variation of jet mass flow with downstream distance showing effect of entrainment.

In spite of the attractiveness of the above arguments there has been increasing evidence that there is a flaw in them somewhere. True, real jets do appear to reach a self-preserving state in which the mean velocity profiles collapse when scaled in *local* variables, say centerline velocity and local half-width. In fact, there is considerable evidence that most, if not all, of these *scaled* profiles look alike, even when source conditions vary. Unfortunately, there is also considerable evidence that the growth rates of jets arising from *different* source conditions are also *different*, a fact that is *not* consistent with the theory which argues that all jets should be asymptotically independent of source conditions.

Clearly there is a problem here! Either the experiments are in error, or the theory is wrong or inapplicable. Certainly it is not easy to make good measurements in free jets because of their sensitivity to external boundary conditions and because of the relatively high local turbulence intensities present (v. Beuther et al. 1987). Nonetheless, it is compelling that even measurements made by single investigators using identical techniques show substantial differences when source conditions are varied (v. Gutmark and Ho 1983).

If the theory is incorrect, why do the measurements in fact confirm a self-preserving state, albeit a multiplicity of them? Is the problem in the plausibility arguments which relate *real* jets to the sources of momentum only? Or are there more fundamental problems in the analysis as originally posed? The answer to both of these questions will be seen below to be yes!

Self-Preservation of Jets: Another Look

We return now to the original formulation of the problem. Without making *any* assumptions about the source conditions, we can reverse the order of the preceding arguments and begin by substituting equations (6)-(9) directly into the governing equations (1)-(3). The results are, with a single exception, identical to equations (17) and (18), i.e.,

$$\left[\frac{\delta}{U_s}\frac{dU_s}{dx}\right]f^2 - \left[\frac{\delta}{U_s}\frac{dU_s}{dx}\right] + 2\left[\frac{d\delta}{dx}\right]\frac{f'}{\eta}\int_0^{\eta} f\,\bar{\eta}d\bar{\eta} \tag{17}$$

$$= \left[\frac{R_s}{U_s^2}\right]\frac{(\eta g_{12})'}{\eta}$$

and

$$\left[U_s^2\delta^2\right]\ 2\pi\int_0^{\infty} f^2\,\eta d\eta \simeq M_o \tag{18}$$

The exception: the functional dependences of U_s, R_s and δ remain to be determined.

From equation (18) it follows immediately that $U_s \sim \delta^{-1}$, say

$$U_s = BM_o^{1/2}\delta^{-1}. \tag{26}$$

Using this, equation (17) reduces to

$$\left[\frac{d\delta}{dx}\right] \left(f^2 + \frac{f'}{\eta}\int_0^{\eta} f\ \bar{\eta}d\bar{\eta}\right) = \left[\frac{R_s}{U_s^2}\right] \frac{(\eta g_{12})'}{\eta} \tag{27}$$

and

$$2\pi \int_0^{\infty} f^2\ \eta d\eta = 1/B^2 \tag{28}$$

Thus the condition for self-preservation reduces to

$$\frac{d\delta}{dx} \sim \frac{R_s}{U_s^2} \tag{29}$$

It is clear that if initial conditions are to enter the problem, they must do so through the quantities U_s, R_s and δ. Thus those arguments which reduced to 'point source of momentum only' solutions must be avoided. Consider, for example, a point source of mass *and* momentum. The scale quantities are now functionally described by,

$$U_s = U_s(x, M_o, m_o) \tag{30a}$$

$$R_s = R_s(x, M_o, m_o) \tag{30b}$$

and

$$\delta = \delta(x, M_o, m_o) \tag{30c}$$

where ρm_o and ρM_o are respectively the rates at which mass and momentum are added. No longer is x the only length scale since a new length scale L can be defined as,

$$L = \frac{m_o}{M_o^{1/2}} \tag{31}$$

On dimensional grounds for our example, solutions which reduce to those for a point source of momentum as $m_o \rightarrow 0$ can be written as,

$$U_s = \frac{M_o^{1/2}}{x} F_1(x/L) \quad (32a)$$

$$R_s = \frac{M_o}{x^2} F_2(x/L) \quad (32b)$$

and

$$\delta = x F_3(x/L) \quad (32c)$$

The plausibility arguments given above for the asymptotic and universal applicability of the point source of momentum solution can now be simply expressed as,

$$\lim_{x/L \to \infty} F_1 \to 1 \quad (33a)$$

$$\lim_{x/L \to \infty} F_2 \to 1 \quad (33b)$$

$$\lim_{x/L \to \infty} F_3 \to A \quad (33c)$$

where $A = d\delta/dx$ must be a universal constant, previously chosen as unity. If this argument is indeed correct, then the substitution of these forms into the equations (27) and (28) should yield differential forms of the x-dependent coefficients, *the solutions to which must be asymptotic to equations (33) regardless of the initial conditions.*

Differentiating equation (32c) and substituting together with equations (32a) and (32b) into the transformed momentum equation yields the condition for self-preservation as,

$$1 + \left[\frac{x}{L}\right] \frac{F_3'}{F_3} \sim \frac{F_2}{F_3 F_1^2} \quad (34)$$

Clearly this equation is satisfied not just for the limiting conditions of equations (33) but by all solutions satisfying,

$$F_3 \sim F_2/F_1^2 \sim (x/L)^m \quad (35)$$

where the exponent m and the constants of proportionality are *arbitrary!* Only one of these (m=o) corresponds to the x-dependence of the 'point source of momentum' solution. Even for this solution there are many *source dependent* possibilities for the constant of proportionality in equation (35) so that different types of jets can grow at different rates. Thus all solutions *diverge* farther from the 'point source of momentum' solution as x increases.

If one abandons even the functional form of the point source of momentum solution as in equations (32a) - (32c), it is easy to see that there are many other classes of functions satisfying the self-preservation constraint of equation (29). In fact, *any x-variation of* $d\delta/dx$ *proportional to the variation of* R_s/U_s^2 *can be self-preserving, at least at the order of the mean momentum equation.* Thus self-preserving jets could spread by any power of x, exponentially, or any other function, at least to this order.

Exponential spreads have been observed by Reynolds and co-workers at Stanford (v. Lee and Reynolds 1985). Even the oft-observed "linear" spread might represent only the first term in an expansion of the actual spreading function. The multiplicity of spreading rates observed by different experimenters might also have been expected since, in principle, no two jets with different starting profiles need be alike. Also, the familiar experiments with "top-hat" jets which are often interpreted as approximations to 'point-source of momentum only' jets, are, in fact, not approximations at all, but have their own unique character. Hence much of the confusion in the literature.

What determines the functional form of $\delta(x)$, and thereby the spreading rate? Alternatively, and somewhat more narrowly, what determines the exponent m? The analysis at this level can provide no insight. What is needed are additional physical constraints, some of which in this case arise from consideration of the second order equations (e.g. kinetic energy) as shown below.

Self-Preservation of Turbulence Quantities

Since the mean momentum equation involves the Reynolds stresses it is reasonable to ask what constraints the equations governing these quantities impose on the self-preserving forms. While equations can be written for each of the Reynolds stresses individually, it will suffice here to examine only the kinetic energy equation.

For the high Reynolds number turbulent jet, the kinetic energy equation reduces to second order to

$$\underline{U \frac{\partial \frac{1}{2}\overline{q^2}}{\partial x}} + V \frac{\partial \frac{1}{2}\overline{q^2}}{\partial r} = \underline{-\frac{1}{r}\frac{\partial}{\partial r} r\left(\overline{pv} + \frac{1}{2}\overline{q^2 v}\right)} - \frac{\partial}{\partial x}\left(\overline{pu} + \frac{1}{2}\overline{q^2 u}\right)$$
$$\underline{-\overline{uv}\frac{\partial U}{\partial r}} - \overline{u^2}\frac{\partial U}{\partial x} - \overline{v^2}\frac{\partial V}{\partial r} \underline{- \epsilon} \qquad (36)$$

where $q^2/2$ represents the kinetic energy per unit mass of the turbulence and ϵ is the rate of dissipation of turbulence energy per unit mass. While all of the terms are important somewhere in the jet, we only need to consider the underlined terms here.

We again allow turbulence and mean quantities to scale independently, and assume self-preserving forms as follows:

$$-\overline{uv} = R_s(x)g_{12}(\eta) \tag{37}$$

$$\frac{1}{2}\overline{q^2} = K_s(x)k(\eta) \tag{38}$$

$$-(\overline{pv} + \frac{1}{2}\overline{q^2v}) = T_s(x)t(\eta) \tag{39}$$

$$\epsilon = D_s(x)e(\epsilon) \tag{40}$$

where K_s, T_s and D_s represent new scale functions for the kinetic energy, third moments and dissipation respectively. Substitution of these equations into equation (36) yields for the underlined terms,

$$\left[U_s K_s{}'\right] fk - \left[U_s \frac{K_s}{\delta}\frac{d\delta}{dx}\right]\eta fk' + \ldots$$

$$= \left[\frac{T_s}{\delta}\right] \frac{(\eta t)'}{\eta} + \ldots \tag{41}$$

$$+ \left[R_s \frac{U_s}{\delta}\right] g_{12}f' + \ldots - \left[D_s\right]e$$

Dividing by $K_s U_s/\delta$ yields,

$$\left[\frac{\delta K_s{}'}{K_s}\right] fk - \left[\frac{d\delta}{dx}\right] \eta fk' + \ldots$$

$$= \left[\frac{T_s}{K_s U_s}\right] \frac{(\eta t)'}{\eta} + \left[\frac{R_s}{K_s}\right] g_{12}f' - \left[\frac{D_s \delta}{K_s U_s}\right]e + \ldots \tag{42}$$

Since for self-preservation, all of the bracketed terms must have the same x-dependence, we must have in particular,

$$\frac{d\delta}{dx} \sim \frac{R_s}{K_s} \tag{43}$$

and

$$\frac{d\delta}{dx} \sim \frac{\delta K_s{}'}{K_s} \tag{44}$$

However, from equation (29), $d\delta/dx \sim R_s/U_s^2$. Therefore equation (43) can be simultaneously satisfied only if

$$K_s \sim U_s^2 \tag{45}$$

Equations (45) and (26) together satisfy equation (44). Thus we are left with

$$\frac{d\delta}{dx} \sim \frac{R_s}{U_s^2} \sim \frac{D_s \delta}{U_s^3} \sim \frac{T_s}{U_s^3} \tag{46}$$

Therefore, the spreading rate of the jet depends on the manner in which the energy dissipation and transport terms scale with respect to the mean motion, not too surprising a result. Note that all of the coefficients remain a function of the initial conditions.

Further reduction of the relationships of equation (46) requires additional equations, or dependence on *ad hoc* assumptions. An obvious choice is to argue that all second order quantities should scale together. Thus we immediately have $R_s = U_s^2$ and linear growth (but at a rate determined by the initial conditions). It will be useful for future reference to look at what happens if instead we choose forms for the dissipation or transport scale functions. For example, for the dissipation we can choose either:

$$\text{Case (i):} \qquad D_s \sim \nu\, U_s{}^2/\delta^2 \tag{47}$$

or

$$\text{Case (ii):} \qquad D_s \sim U_s{}^3/\delta \tag{48}$$

We are accustomed (v. Tennekes and Lumley 1972) to associate the first with low Reynolds number turbulence where the energy-containing eddies are affected directly by viscosity, and the second with high Reynolds number turbulence where viscosity has no direct effect on the energy containing scales.

The results are,

$$\text{Case (i):} \qquad \frac{d\delta}{dx} \sim \frac{R_s}{U_s{}^2} \sim \frac{T_s}{U_s{}^3} \sim \nu/M_o{}^{1/2} = \text{constant} \tag{49a}$$

and

$$\text{Case (ii):} \qquad \frac{d\delta}{dx} \sim \frac{R_s}{U_s{}^2} \sim \frac{T_s}{U_s{}^3} \sim \text{constant} \tag{49b}$$

where the constants in <u>both</u> cases depend on the details of the initial conditions. The two scenarios differ, however, in that for case (i) the spreading rate (as measured by $d\delta/dx$) varies inversely with the initial Reynolds number, $M_o{}^{1/2}/\nu$, whereas for case (ii) the Reynolds number dependence (if any) is more subtle. This latter weak dependence appears to be more consistent with the experimental observations in high Reynolds number jets, and would appear to confirm, for now, the conventional wisdom regarding the dissipation.

Thus for any of the above assumptions we have

$$\delta \sim x \tag{50a}$$

$$R_s \sim U_s^2 \tag{50b}$$

$$T_s \sim U_s^3 \tag{50c}$$

and

$$D_s \sim U_s^3/x \tag{50d}$$

the latter resulting from the constancy of the Reynolds number $U_s\delta/\nu$ as the flow develops. Therefore self-preservation to the second-order is possible, and the flow can, in fact, be characterized by only a single length and velocity scale, the *assumption* of the traditional analyses. It is important to recognize, however, that the constraints derived above do *not* require that all jets look like the hypothetical 'point source of momentum only' jet. *Nor do they even require that all jets grow and decay at the same rates!* This is because that even though the coefficients of equations (27), (28) and (42) are x-independent, they can still depend on the details of how the flow is generated.

It is important to ask whether jets with differing growth rates can have different profiles for f, g_{12}, k, etc.? This question can be answered by examining the momentum equation (27) which can be reduced to

$$-f^2 + \frac{f'}{\eta} \int_0^\eta f\ \bar{\eta} d\bar{\eta} = \left[\frac{R_s}{U_s^2} \left(\frac{d\delta}{dx}\right)^{-1}\right] \frac{(\eta g_{12})'}{\eta} \tag{51}$$

The term in brackets is now x-independent *but can vary from flow to flow!* Therefore *all* self-preserving jets will have the same velocity profile *shape*, $f(\eta)$, and the same Reynolds stress profile *shape*, $g_{12}(\eta)$. The scale factors for these quantities may differ, however, so that depending on how the profiles are normalized they may differ by an amplitude factor. For example, if the mean velocity and Reynolds stress are normalized by centerline velocity and centerline velocity squared respectively, the mean velocity profile will look the same for all jets. The Reynolds stress profiles will the same shape, but will have an amplitude factor which varies from jet to jet because of the factor $[(R_s/U_s^2)/(d\delta/dx)]$ which has been absorbed into it.

While the mean velocity and Reynolds stress profiles must maintain the same relative shape from flow to flow, this is *not* the case for the other turbulence quantities. This is clear from the more complicated nature of equation (42) in which the coefficients are determined by the initial conditions in a way which allows the terms to differ *relative* to each other, resulting in different solutions for each flow. Thus, for example, a top-hat jet should not be expected to have the same profile of kinetic energy as one starting from a fully developed pipe flow, even though their mean velocity and Reynolds stress profiles will have the same shape.

Finally it should be clear from the conditions for self-preservation of equations (46) and the subsequent discussion that there may exist a variety of ways in which the jet can develop in a self-preserving manner. The linear spreading rate has been seen to result when all of the moments 'settle-in' together. It may be possible to modulate this development at the source by excitation so that all of the flow quantities vary in a self-preserving (and possibly exotic) manner. It might be interesting to re-examine some of the experiments on coherent structures in forced and non-circular jets from this perspective.

The experiments of Binder and Favre-Martinet (1981) on forced axisymmetric and two-dimensional jets illustrate both the theory presented here and the possibilities for future work. While the forcing affected only the virtual origin of the axisymmetric jet, it dramatically affected the spreading rate for the planar jet. The scaled mean velocity profiles for each case, however, remained the same, regardless of forcing. The analysis can easily be extended to be two-dimensional jet with equivalent results -- a dependence of spreading rate on initial conditions, but with profile shapes independent of them. Figure (3) shows the effect of the initial conditions on the variation with distance of the centerline mean velocity for the two-dimensional jet.

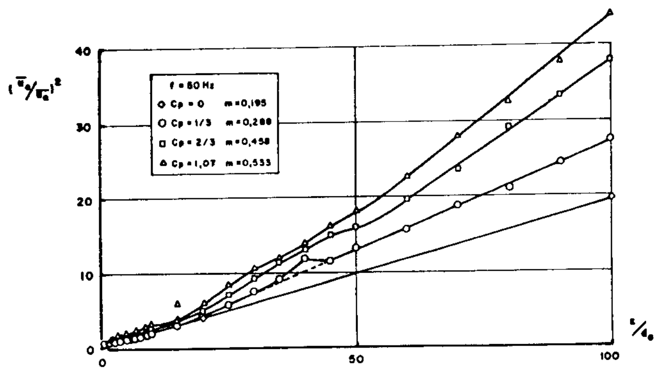

Figure 3. Variation of two-dimensional jet centerline velocity with downstream distance showing effect of forcing. (from Binder and Favre-Martinet 1981). Theory predicts inverse square root dependence for full self-preservation.

The Plane Wake

Another turbulent free shear flow which has been extensively investigated is the turbulent far-wake of a drag-producing obstacle as illustrated in Figure (4). Recent experiments by Wygnanski et al. (1986) show clearly that the wake characteristics are not universal as expected from the classical analysis (v. Sreenivasan and Narashima, 1982). In the following paragraphs we shall briefly outline the analysis for a plane wake and show that, as for the jet, the observed features are consistent with self-preserving motions which are determined by their initial conditions.

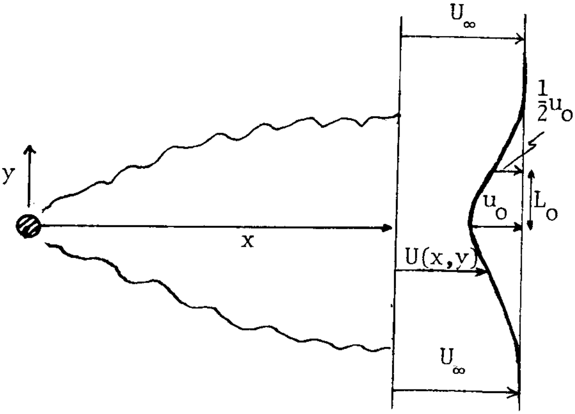

Figure 4. Sketch showing development of wake behind drag-producing obstacle.

The governing equation for a plane wake in a uniform stream to *first order* are (Townsend 1976),

$$U_\infty \frac{\partial}{\partial x} (U-U_\infty) = - \frac{\partial}{\partial y} \overline{uv} \tag{52}$$

where U_∞ is the speed of the free stream. The momentum integral to first order is given by

$$\frac{d}{dx} \int_{-\infty}^{\infty} U_\infty(U-U_\infty)dy = 0 \tag{53}$$

which can be integrated to yield

$$\int_{-\infty}^{\infty} U_\infty(U-U_\infty)dy = U_\infty^2 \ \theta = M/\rho \tag{54}$$

where θ is defined to be the constant momentum thickness and M is the momentum defect per unit length on the wake generator.

Self-preserving solutions are sought of the following forms:

$$U-U_\infty = U_s(x)f(\eta) \tag{55}$$

$$-\overline{uv} = R_s(x)g_{12}(\eta) \tag{56}$$

and

$$\eta = r/\delta \quad \text{where} \quad \delta = \delta(x) \tag{57}$$

These can be substituted directly into equation (52) to yield

$$\left[U_\infty \frac{dU_s}{dx}\right] f - \left[U_\infty \frac{U_s}{\delta} \frac{d\delta}{dx}\right] \eta f' = \left[\frac{R_s}{\delta}\right] g_{12} \qquad (58a)$$

which after rearranging can be written as

$$\left[\frac{\delta}{U_s} \frac{dU_s}{dx}\right] f - \left[\frac{d\delta}{dx}\right] \eta f' = \left[\frac{R_s}{U_\infty U_s}\right] g_{12}' \qquad (58b)$$

Similarly, equation (54) becomes,

$$\left[U_s U_\infty \delta\right] \int_{-\infty}^{\infty} f \, d\eta = U_\infty^2 \, \theta = \text{const} \qquad (59)$$

Since all of the x-dependence is in the bracketed terms and since the right-hand side of equation (59) is constant, we must have

$$U_s/U_\infty \sim \theta/\delta \qquad (60a)$$

or

$$U_s/U_\infty = B \; \theta/\delta \qquad (60b)$$

where B is the constant of proportionality. Thus equation (58) reduces to

$$-\left[\frac{d\delta}{dx}\right] (f+\eta f') = \left[\frac{R_s}{U_\infty U_s}\right] g_{12}' \qquad (61)$$

Therefore any δ, R_s and U_s satisfying

$$\frac{d\delta}{dx} \sim \frac{R_s}{U_\infty U_s} \qquad (62)$$

represents a possible self-preserving solution of the momentum equation.

Equation (61) can be integrated to yield

$$-\eta f = \left[\frac{R_s}{U_\infty U_s} \left(\frac{d\delta}{dx}\right)^{-1}\right] g_{12} \tag{63}$$

Thus self-preservation requires the equivalent condition,

$$\frac{R_s}{U_\infty U_s} \left(\frac{d\delta}{dx}\right)^{-1} \sim \text{constant} \tag{64}$$

The Point Source of Drag

The traditional analysis supposes that the wake is generated by a point source of drag only, and is therefore completely characterized by the kinematic drag, $U_\infty^2\theta$, and the distance downstream, x. Thus on dimensional grounds,

$$U_s = U_s(x, U_\infty^2\theta) \tag{65}$$

and

$$\delta = \delta(x, U_\infty^2\theta) \tag{66}$$

subject to the momentum integral constraint,

$$U_s\delta \sim U_\infty\theta \tag{60}$$

From equation (65), the only possibility on dimensional grounds is

$$U_s \sim \left[\frac{U_\infty^2\theta}{x}\right]^{1/2} \tag{67}$$

Therefore from (67) we must have,

$$\delta \sim (\theta x)^{1/2} \tag{68}$$

It has been commonly believed that all plane wakes, regardless of generator, should asymptotically reduce to this single and universal 'point source of drag' wake. It is easy to see, however, that (like the jet) as soon as additional parameters arising from the details of the initial conditions are allowed into the dimensional analysis statements of equations (65) and (66), a considerably wider variety of possible solutions are possible. Each of these solutions will retain in some way the features imparted indelibly to it by its initial conditions. This will be discussed in further detail below.

WAKES GENERATED BY REAL SOURCES

It is clear from equation (63) that variety of self-preserving wake profiles are possible as long as equation (62) is satisfied. As was the case for the jet, only some of these profiles are consistent with the self-preservation of second order quantities. To see this consider the equation for the kinetic energy balance given to first order by

$$U_\infty \frac{\partial \frac{1}{2}\overline{q^2}}{\partial x} = \frac{\partial}{\partial y}\left[-\frac{1}{2}\overline{q^2 v} - \frac{\overline{pv}}{\rho}\right] - \overline{uv}\frac{\partial U}{\partial y} - \epsilon \tag{69}$$

Defining,

$$\frac{1}{2}\overline{q^2} = K_s(x)k(\eta) \tag{70a}$$

$$-\frac{1}{2}\overline{q^2 v} - \frac{\overline{pv}}{\rho} = T_s(x)t(\eta) \tag{70b}$$

$$\epsilon = D_s(x)e(\eta), \tag{70c}$$

and using equations (55)-(57), yields the following transformed equation:

$$\left[U_\infty \frac{dK_s}{dx}\right] k - \left[U_\infty \frac{K_s}{\delta}\frac{d\delta}{dx}\right]\eta k' = \left[\frac{T_s}{\delta}\right]t' - \left[\frac{R_s U_s}{\delta}\right]g_{12}f' - \left[D_s\right]e \tag{71a}$$

or equivalently,

$$\left[\frac{\delta}{K_s}\frac{dK_s}{dx}\right]k - \left[\frac{d\delta}{dx}\right]\eta k' = \left[\frac{T_s}{K_s U_\infty}\right]t' - \left[\left(\frac{U_s}{U_\infty}\right)\frac{R_s}{K_s}\right] g_{12}f' - \left[\frac{D_s\delta}{K_s U_\infty}\right]e \tag{71b}$$

Thus self-preservation of the second order quantities requires,

$$\frac{d\delta}{dx} \sim \frac{\delta}{K_s}\frac{dK_s}{dx} \tag{72a}$$

$$\frac{d\delta}{dx} \sim \left[\frac{U_s}{U_\infty}\right]\frac{R_s}{K_s} \tag{72b}$$

and

$$\frac{d\delta}{dx} \sim \left[\frac{U_s}{U_\infty}\right]\frac{T_s}{K_s U_s} \sim \left[\frac{U_s}{U_\infty}\right]\frac{D_s}{K_s U_s} \tag{72c}$$

Equations (62) and (72b) lead immediately to the conclusion that

$$K_s \sim U_s^2 \tag{73}$$

from which it follows from equations (60) and (72c) that

$$\frac{d\delta}{dx} \sim \left[\frac{\theta}{\delta}\right] \frac{R_s}{K_s} \sim \left(\frac{\theta}{\delta}\right) \frac{T_s}{U_s^3} \sim \left[\frac{\theta}{\delta}\right] \frac{D_s \delta}{U_s^3} \tag{74}$$

Again we are confronted with the need for additional equations or *ad hoc* assumptions. We can assume $R_s \sim K_s$, $D_s \sim U_s^3/\delta$ or $D_s \sim \nu U_s^2/\delta^2$, and all lead to the same x-dependence (because $U_\infty \theta/\nu$ is constant from the momentum constraint). Thus, $d\delta/dx \sim \theta/\delta$ from which it follows that

$$\delta^2 \sim \theta x \tag{75}$$

(Note that as for the jet, the latter assumption for D_s yields a growth rate which depends inversely on Reynolds number.)

Thus, even without the restriction to a point source of drag, the self-preserving wake must grow as

$$\delta \sim (\theta x)^{1/2} \tag{76}$$

From equation (62), it follows that the velocity scale function, U_s, is given by

$$U_s \sim U_\infty/(x/\theta)^{1/2} \tag{77}$$

Note that the coefficients in equations (75) and (77) are not uniquely determined by the analysis (as for the point source wake) but depend on the spreading rate (and thus on the initial conditions).

There are other interesting consequences of the dependence of the wake development on initial conditions. Examination of equation (63) reveals that the profiles of mean velocity and Reynolds stress (when scaled with local velocity deficit and width) will be the same for all wakes (to within a constant factor determined by $(R_s/U_s^2/d\delta/dx)$ Thus, like the jet, even though different wakes can spread at very different rates, their scaled mean velocity and Reynolds stress profiles will be similar in shape. If the centerline velocity deficit is used as the normalizing parameter, all wakes will even have the same normalized profiles. The Reynolds stress profiles, however, will differ by a scale factor. On the other hand, the kinetic energy equation given by equation (71) depends in a non-simple manner on the growth and decay constants. Therefore the profiles of kinetic energy can, in principle, vary from wake to wake, even though each is self-preserving!

Strong experimental evidence for the above conclusions is provided by the recent experiments of Wygnanski et al. (1986) who examined in detail the asymptotic character of three very different wake generators. Although the normalized mean velocity profiles were identical, the variation of centerline velocity deficit with distance and the spreading rates showed a strong dependence on the initial conditions (Figure 5). As predicted, the Reynolds stress profiles (Figure 6) had the same shape but with an amplitude factor determined by the spreading rate (since momentum was conserved). The turbulence intensity profiles were distinctly different for the various generators (Figure 7).

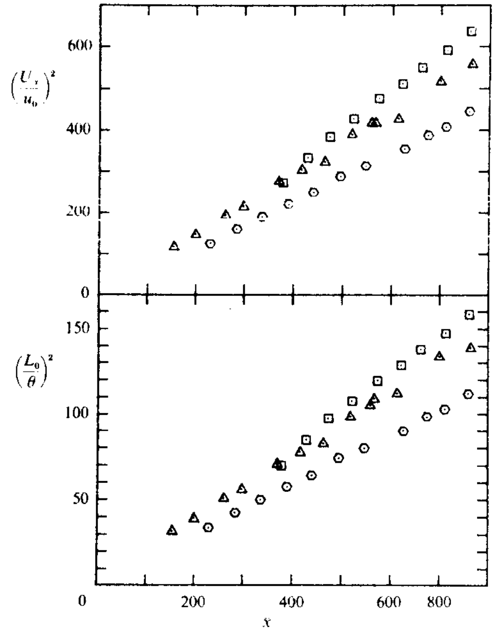

Figure 5. Variation centerline velocity deficit and half-width with distance for three wake generators; □ ,airfoil; △, 70% solidity screen; ○ , solid strip (from Wygnanski et al. 1986).

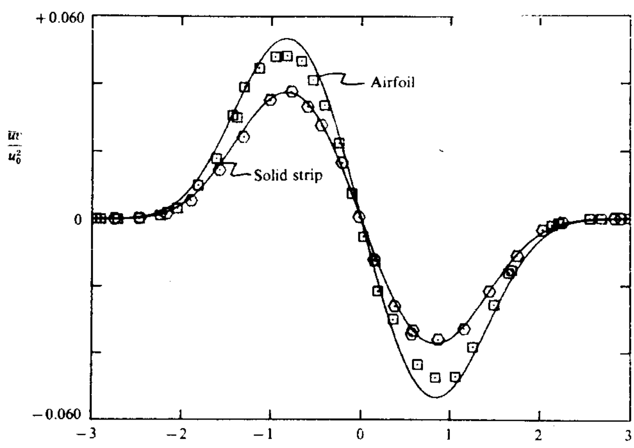

Figure 6. Reynolds stress normalized by centerline velocity deficit for the solid strip and airfoil (Wygnanski et al. 1986).

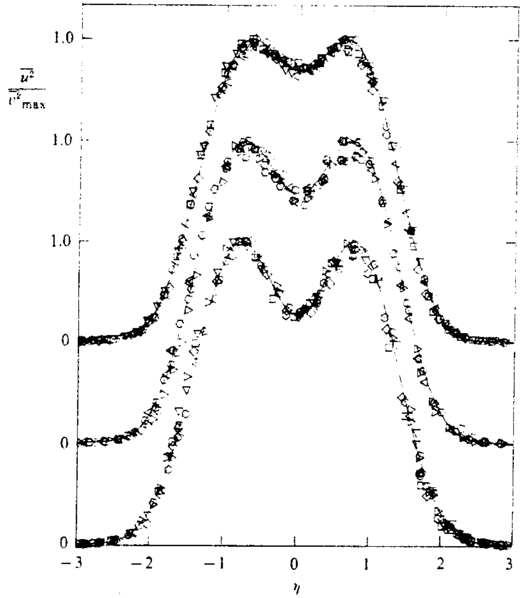

Figure 7. Normalized turbulence intensity profiles for the three generators of Figure 5 (Wygnanski et al. 1986).

THE AXISYMMETRIC WAKE: A FLOW WHICH DOES NOT EVOLVE AT CONSTANT REYNOLDS NUMBER

The axisymmetric wake presents an interesting contrast to the axisymmetric jet and plane wake flows described above in that it does not evolve at constant Reynolds number (as will be seen). As a consequence, the nature of the assumptions regarding the dissipation will be seen to predict two quite different asymptotic developments. There appears to be experimental evidence for both forms in different experiments, which raises an interesting question as to how the flow chooses one form or another. An interesting possibility is that the flow evolves from one state to another as the Reynolds number changes. These possibilities will be discussed in more detail below following the analysis.

The equations of motion describing the axisymmetric wake to first order can be shown to reduce to,

$$U_\infty \frac{\partial}{\partial x}\left[U-U_\infty\right] = -\frac{1}{r}\frac{\partial}{\partial r} r\,\overline{uv} \tag{78}$$

where U_∞ is the undisturbed speed of the free stream. This can be integrated across the flow to yield the integral constraint,

$$2\pi \int_0^\infty U_\infty\,(U-U_\infty)r\,dr = \pi U_\infty{}^2\theta^2 = M/\rho \tag{79}$$

where θ is defined to be the momentum thickness.

As before we seek self-preserving solutions of the form

$$U-U_\infty = U_s f(\eta) \tag{80}$$

$$-\overline{uv} = R_s \; g_{12}(\eta) \tag{81}$$

where $$\eta = r/\delta \tag{82}$$

and $$\delta = \delta(x) \tag{83}$$

Substituting equations (80) – (82) into the governing equation (78) yields after some manipulation,

$$\left[\frac{\delta}{U_s}\frac{dU_s}{dx}\right] f - \left[\frac{d\delta}{dx}\right]\eta f' = \left[\frac{R_s}{U_\infty U_s}\right]\frac{1}{\eta}\frac{d}{d\eta}\left[\eta g_{12}\right] \tag{84}$$

For self-preservation, all of the bracketed terms must have the same x-dependence so that

$$\frac{d\delta}{dx} \sim \frac{\delta}{U_s}\frac{dU_s}{dx} \tag{85}$$

and

$$\frac{d\delta}{dx} \sim \frac{R_s}{U_\infty U_s} \tag{86}$$

The integral constraint, equation (79), can be transformed to yield

$$\left[U_\infty U_s \delta^2\right] 2\pi \int_0^\infty f\eta d\eta = \pi U_\infty{}^2 \theta^2 \tag{87}$$

It follows immediately from equation (87) that

$$U_s \delta^2 \sim U_\infty \theta^2 \tag{88a}$$

or

$$U_s/U_\infty \sim (\theta/\delta)^2 \tag{88b}$$

Therefore equation (85) is satisfied identically, leaving only equation (86) to be satisfied for self-preservation.

The momentum equation can now be written as

$$- \left[\frac{d\delta}{dx}\right] \left[2\eta f + \eta^2 f'\right] = \left[\frac{R_s}{U_\infty U_s}\right] \frac{d}{d\eta} \left[\eta g_{12}\right] \tag{89}$$

which can be reduced to

$$- \left[\eta^2 f\right]' = \left[\frac{R_s}{U_\infty U_s}\left[\frac{d\delta}{dx}\right]^{-1}\right]\frac{d}{d\eta} \left[\eta g_{12}\right] \tag{90}$$

This can in turn be integrated to yield

$$- \eta f = \left[\frac{R_s}{U_\infty U_s}\left[\frac{d\delta}{dx}\right]^{-1}\right] g_{12} \tag{91}$$

It is immediately clear, to this point at least, that the axisymmetric wake will have features like those observed for the other flows described above. In particular, the velocity profiles will look alike for all wake generators when scaled by the centerline velocity, and the Reynolds stress profile shapes will be the same but differ by an amplitude factor of $[R_s/(U_\infty U_s)]/(d\delta/dx)$.

As before we turn to the kinetic energy equation to examine the possibility of full self-preservation. For the axisymmetric wake, this can be written to first order as

$$U_\infty \frac{\partial \frac{1}{2} \overline{q^2}}{\partial x} = - \overline{uv} \frac{\partial U}{\partial r} - \frac{1}{r} \frac{\partial}{\partial r} r \left[\frac{1}{2} \overline{q^2 v} + \frac{\overline{pv}}{\rho}\right] - \epsilon \tag{92}$$

Three new scaling functions are needed; in particular

$$\frac{1}{2} \overline{q^2} = K_s(x) k(\eta) \tag{93a}$$

$$\epsilon = D_s(x) e(\eta) \tag{93b}$$

and

$$\left[- \frac{1}{2} \overline{q^2 v} + \frac{\overline{pv}}{\rho}\right] = T_s(x) t(\eta) \tag{93c}$$

These can be substituted into equation (92) to yield

$$\left[\frac{\delta}{K_s} \frac{dK_s}{dx}\right] k - \left[\frac{d\delta}{dx}\right] \eta k' = \left[\frac{R_s}{K_s}\left[\frac{U_s}{U_\infty}\right]\right] g_{12} f' + \left[\frac{T_s}{K_s U_\infty}\right] t - \left[\frac{D_s \delta}{K_s U_\infty}\right] e \tag{94}$$

Thus for self-preservation to this order we require:

$$\frac{d\delta}{dx} \sim \frac{\delta}{K_s}\frac{dK_s}{dx} \tag{95}$$

$$\frac{d\delta}{dx} \sim \frac{R_s}{K_s}\left[\frac{U_s}{U_\infty}\right] \tag{96}$$

$$\frac{d\delta}{dx} \sim \frac{T_s}{K_s U_\infty} \tag{97}$$

and

$$\frac{d\delta}{dx} \sim \frac{D_s \delta}{K_s U_\infty} \tag{98}$$

From equations (86) and (96) it follows that

$$K_s \sim U_s{}^2 \tag{99}$$

with the consequence that equation (95) reduces to

$$\frac{\delta}{K_s}\frac{dK_s}{dx} \sim \frac{\delta}{U_s}\frac{dU_s}{dx} \sim - \frac{d\delta}{dx} \tag{100}$$

using equation (88). Similarly equation (96) reduces to exactly equation (86). Therefore, for self-preservation,

$$\frac{d\delta}{dx} \sim \frac{R_s}{U_s U_\infty} \sim \frac{D_s \delta}{U_s^2 U_\infty} \sim \frac{T_s}{U_s^2 U_\infty} \tag{101}$$

It is again apparent that we must either introduce additional equations or *ad hoc* assumptions to proceed further. As before we try assuming the form of the dissipation scale function, D_s. We consider two cases:

$$\text{Case (i):} \qquad D_s \sim U_s^3/\delta \tag{102}$$

$$\text{Case (ii):} \qquad D_s \sim \nu U_s^2/\delta^2 \tag{103}$$

(Note that Case (i) includes $R_s \sim K_s$ from equation (101) since $K_s \sim U_s^2$.)

For Case (i) it is easy to show from equations (88), (101) and (102) that

$$\frac{d\delta}{dx} \sim \frac{U_s}{U_\infty} \sim \left[\frac{\theta}{\delta}\right]^2 \tag{104}$$

Thus

$$\delta^3 \sim \theta^2 x$$

or

$$\text{Case (i):} \qquad \delta/\theta \sim (x/\theta)^{1/3} \tag{105}$$

It follows from equation (88) that

$$\text{Case (i):} \qquad \frac{U_s}{U_\infty} \sim \left[\frac{\theta}{x}\right]^{2/3} \tag{106}$$

The Reynolds number governing the downstream development of the flow is given by

$$\text{Case (i):} \qquad U_s\delta/\nu \sim \frac{U_\infty\theta}{\nu}\left[\frac{\theta}{x}\right]^{1/3} \tag{107}$$

and thus decreases with downstream distance.

Equations (105) and (106) represent the classical results for the axisymmetric wake (v. Tennekes and Lumley 1972) except that now the coefficients will be determined by the initial conditions as demonstrated for the jet and plane wake flows earlier.

Now consider Case (ii) for which it can be shown from equations (101) and (103) that

$$\frac{d\delta}{dx} \sim \frac{\nu}{U_\infty\delta} \tag{108}$$

This can be integrated to yield

$$\delta^2 \sim \frac{\nu x}{U_\infty}$$

or

$$\text{Case (ii)} \qquad \left[\delta/\theta\right] \sim \left[\frac{\nu}{U_\infty\theta}\right]^{1/2}\left[\frac{x}{\theta}\right]^{1/2} \tag{109}$$

Thus, unlike Case (i), where the wake spreads as the one-third power of x, this wake spreads as <u>the square root of x, and the rate of spread is directly proportional to the inverse square root of the Reynolds number.</u> It follows from equation (88) that

$$\text{Case (ii)} \qquad \frac{U_s}{U_\infty} \sim \left[\frac{U_\infty\theta}{\nu}\right]\left[\frac{\theta}{x}\right] \tag{110}$$

The Reynolds number governing the downstream development is now given by

$$\text{Case (ii)} \qquad U_s\delta/\nu \sim \left[\frac{U_\infty\theta}{\nu}\right]^{3/2}\left[\frac{\theta}{x}\right]^{1/2} \sim \text{constant} \qquad (111)$$

which decreases even faster than for Case (i) with increasing x.

If the traditional high Reynolds number arguments about energy transfer are correct then Case (i) would govern during an initial period of self-preservation until the Reynolds number is sufficiently reduced for Case (ii) take over. Thus for high Reynolds number axisymmetric wake, both cube root and square root regimes should be observed in succession.

An alternate possibility, which defies the conventional wisdom, is that the flow may organize itself so that the development proceeds at the highest possible Reynolds number. If this were the case then the order would be reversed since the Reynolds number for Case (ii) depends on a higher power of $U_\infty\theta/\nu$ than does Case (i) until x/θ exceeds a certain value. It is easy to show from equations (107) and (111) that this value is proportional to

$$\left[\frac{x}{\theta}\right]_{crit} \sim \left[\frac{U_\infty\theta}{\nu}\right]^{3} \qquad (112)$$

For any reasonable value of the Reynolds number, $U_\infty\theta/\nu$, this point is orders of magnitude beyond the range of experiment. Thus only the square root type wake evolution should be observed.

The experimental evidence at the point is not yet definitive. The recent experiments of Cannon, Champagne and Wygnanski (1987) exhibit the square root behavior corresponding to Case (ii), unlike earlier measurements which appeared to show the cube root behavior. Clearly there is work yet to be done which could have important consequences if the more recent measurements are sustained since the traditional line of reasoning would appear to give an incorrect result.

Self-Preservation Analysis of Other Turbulent Flows

It is clear from the preceding examples that much of what has been believed about turbulent free shear flows needs to be reexamined. At the very least the classical similarity analyses for other flows must be re-interpreted since these flows must also be forever influenced by their initial conditions, just as the jet and the wake are. For example, in view of the analysis presented here, one simply cannot expect the mixing layer between two streams of differing speeds to be independent of the characteristics of the splitter plate behind which it develops. Similarily, the analyses of Fisher et al. (1979) and Baker et al. (1982) for the manner in which hot jets develop asymptotically into buoyant plumes must also be reinterpreted since the asymptotic states cannot be independent of the manner in which the flow is generated.

In addition, there are probably many flows which are self-preserving at the first order, but not at the second. A common characteristic of such flows will be the non-linear spreading rates and a Reynolds stress scale which does not equal the square of the velocity scale, but for which $(R_s/U_s{}^2)/(d\delta/dx)$ or its equivalent is a constant. Note that the traditional plot of $\overline{uv}/U_s^2$ will mask this kind of first order only self-preservation since the Reynolds stress will have its own scale. A better choice for normalization is $\overline{uv}/(U_s^2 d\delta/dx)$, or even simply normalizing the Reynolds stress by its peak value.

Self-Preservation and Kolmogorov's Theories for Small Scale Turbulence

The re-examination of self-preserving flows previously thought to be well understood can have startling and important consequences. George (1987a) has reconsidered the analysis of von Karman and Howarth (1938) and the subsequent analysis of Batchelor (1948) for the self-preservation of the energy spectrum of isotropic decaying turbulence. Contrary to these earlier analyses, it was found to be possible to collapse the spectrum *at all scales of motion* at all Reynolds numbers with a single length scale. The characteristic length scale was found to be the Taylor microscale and the characteristic velocity scale was determined by the energy. Experimental confirmation was provided by the careful experiments of Comte-Bellot and Corrsin (1971) as illustrated in Figure (8). A consequence of the analysis was that Kolmogorov's theories (Kolmogorov 1941) for the universal equilibrium range could be shown to be incorrect when applied to this important flow.

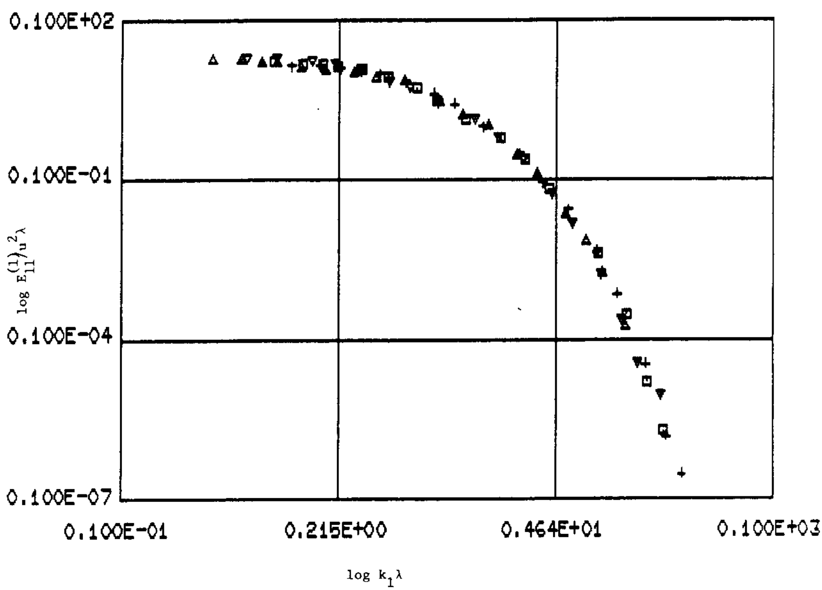

Figure 8. Spectra measured behind one-inch grid normalized using turbulence intensity and Taylor microscale: Δ,x/m=45; ∇,120; □ ,240; +,385 (from George 1987a)

Similar considerations of self-preservation at all scales of motion have since been extended to a number of homogeneous turbulent flows including the homogeneous shear flow and homogeneous strain experiments (v. George 1987b, 1988, George and Gibson 1988). All were found to be fully self-preserving at all scales of motion, and to be inconsistent with Kolmogorov's arguments.

An explanation for the failure of many homogeneous flows to scale according to Kolmogorov's theory of local similarity of the small scale motions is that the flow has answered to a higher calling: that of full self-preservation at all scales of motion. This is precisely Conjecture I. That this might be a phenomenon present in other flows is illustrated by the spectral data of Champagne (1978) reproduced in Figure (9). Clearly the spectra for the wake, jet and atmosphere are different at high wavenumbers, even though normalized by Kolmogorov variables. Champagne attributes this to the Reynolds number dependence of the universal spectral form.

An alternate explanation becomes apparent if one considers that both the jet and the wake, unlike the atmosphere, can be truly self-preserving *at all scales of motion* since they evolve at constant Reynolds number, $U_s\delta/\nu$. As a consequence, all of the characteristic length scales -- $\nu U^2/\epsilon$, $\overline{q^2}^{2}/\epsilon$, $(\nu^3/\epsilon)^{1/4}$ where ϵ is the rate of dissipation, $\overline{q^2}$ is the kinetic energy, and ν is the kinematic viscosity --are proportional and each will equally collapse the spectral data at all scales of motion. More importantly, as was shown earlier for the averaged equations and can also be shown for the spectral equations, the spectral shapes can depend on the initial conditions. Thus the jet and the wake spectra can not *a priori* be expected to have the same characteristics, even at the highest wavenumbers. Each flow may simply have responded to its own governing equations and initial conditions by relaxing into its own unique spectral shape. The Reynolds number variation observed by Champagne may be simply a measure of the fact that the flows themselves were different, as well as confirmation that the high wavenumber spectrum is not universal.

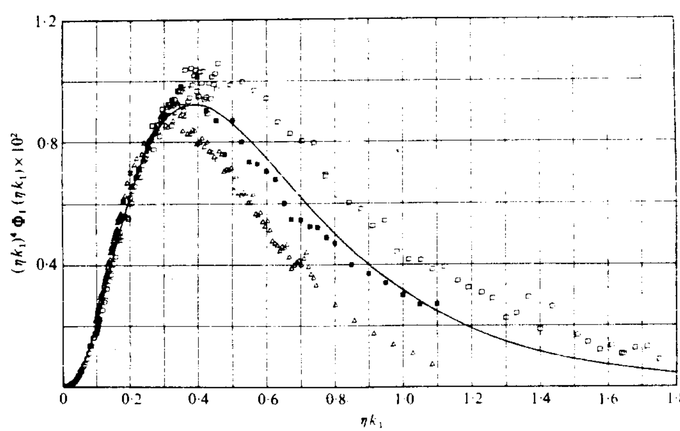

Figure 9. Fourth moments of normalized spectra: □, jet; —, corrected jet; Δ, cylinder wake; ▲, boundary layer (Champagne 1978).

It should be noted that these ideas are not far afield from the modified Kolmogorov theory proposed in 1963 (Kolmogorov 1963) which attempts to account for the spatially localized nature of the dissipation in turbulent flows. Self-preservation, perhaps through the role of the large scale structures (as suggested below), has determined the distribution of the dissipation so that the type of flow (along with the Reynolds number) is the governing characteristic. This is consistent with observed failure of general assumptions about the distribution of dissipation, like log normality (v. Wyngaard and Tennekes 1970), as well as the persistent evidence of the non-isotropy of the dissipation in self-preserving flows (v. Browne et al. 1987).

The apparent success of Kolmogorov's theories in flows like the atmosphere (Pond et al. 1963) and the tidal channel (Grant et al. 1962) is not inconsistent with the ideas put forth above. According to Conjecture II: When the equations of motion and boundary conditions do not admit to self-preserving solutions, the flow will adjust itself as closely as possible to a self-preserving state. Thus, in the absence of a single length scale, the turbulence will assume a locally self-preserving character. Whether such local self-preservation can be universal (in the sense that the constants are the same from flow to flow) or whether it must always retain a dependence on the larger scales and mean flow is open to question. The hypothesis put forth above would argue that the constants (like Kolmogorov's and von Karman's) must always retain a dependence on the particular flow and thus cannot assume universal values. If so, then a number of questions assumed answered need to be reexamined. Examples include the behavior of spectra in the equilibrium range and the nature of flows next to walls, to cite but a few.

COHERENT STRUCTURES AND SELF-PRESERVATION

Another subject which must be closely related to the analysis here is that of coherent structures. It has long been argued that initial conditions are important in determining the development of such structures in shear flows, and that these structures in turn influence the spreading rates of the flows. That this is true has been extensively documented (e.g. Gutmark and Ho 1983, Hussain 1983), but never explained. Whatever the explanation for the detailed behavior of a particular structure, however, the general characteristics are consistent with the apparent tendency of turbulent (and non-turbulent) flows to settle rapidly into a self-preserving state (if permitted to do so by the dynamical equations), and thereby to remember forever how they began. Thus the basis for Conjectures I and II.

An example as to what this relationship between coherent structures and self-preservation might be is possibly given by the recent model proposed by Glauser and George (1987) for the dynamics of an axisymmetric shear layer. In brief, ring-like concentrations of vorticity arise from instabilities of the mean profile which then interact to initiate a sequence of instabilities which are responsible for the "cascade" of energy to small scales. The key point of importance here is that each new instability is triggered by the preceding one, and occurs in a flow modified by it. It is not difficult to imagine how such a process could be forever influenced by the manner in which it is initiated, and why it might show a preference for being governed by equations whose terms are (on the average) in relative equilibrium with respect to each other.

Another example comes from the self-preservation analysis of grid turbulence mentioned above. George (1987a) has suggested that such self-preserving behavior for the decay of turbulence behind a grid might be due simply to the evolution of single vortical structures distributed randomly throughout the flow. The manner in which grid turbulence changes into the final period of decay (as characterized by the $t^{-5/2}$ decay of the energy) has been shown to be consistent with evolution of the flow from one self-preserving state to another in which (after the evolution) the tails of the correlations roll-off faster than r^{-5}. This evolution is also consistent with an instability and breakdown of the original vortical structures.

Self-Preservation and Turbulence Modeling

That insight into the dynamics of coherent structures is important to future developments of turbulence becomes immediately apparent by considering another consequence of the results presented earlier. *None of the existing single point closures models of turbulence now so prevalently used by the engineering community can account for this non-universal development of self-preserving flows.* While this problem is discussed in some detail by Taulbee (1988), it is worthwhile to briefly consider it here.

All turbulence models make closure approximations which relate higher order moments to lower order ones. Single point models at some level assume that these relations are locally determined, and are independent of history. As a consequence, only a single asymptotic state can be predicted which is independent of initial conditions; in particular, the state determined by the model constants. The phenomenon was first documented by Taulbee and Lumley (1981) who used a Reynolds stress model to successfully predict the properties of the far-wake of a screen, but were unable to account for the observations behind cylinders. As an alternative to suggesting that the measurements were wrong (this was before the decisive measurements by Wygnanski et al. (1986) mentioned earlier), they suggested that the problem might lie in the presence of coherent structures in the cylinder far-wake which were not present in the screen far-wake.

The problem with the turbulence models, in view of the arguments presented here, can be stated simply as follows: *A turbulence model (single point, at least) can predict the asymptotic development of self-preserving free shear flows only by selecting a set of "universal" constants appropriate to the particular initial conditions.* Note that this is far worse than simply needing different constants for axisymmetric and plane flows since each of these flows will require a multiplicity of constants to account for different starting conditions. While perhaps acceptable in some engineering situations, this is clearly intolerable to the theoretician and indicates that there is considerable work to be done. Most probably, progress will come about as suggested above (and by Taulbee and Lumley) by addressing the dynamical role of the coherent structures, a subject which is only beginning to be addressed by the theoreticians.

Acknowledgments

The ideas presented in this paper began in the Spring of 1986 with a series of lectures on turbulence to graduate students which did not go as planned. On the day I had intended to begin the study of self-preservation of free shear flows, I was involved in a meeting which ran so late that I had to choose between arriving at my class on time or returning to my office to retrieve my lecture notes. With the confidence of having given similar lectures many times, I opted to lecture without notes. Forgetting that I should first justify by heuristic arguments and dimensional analysis the use of a single length and velocity scale, I plunged immediately into an analysis of the equations, and thus by accident discovered a wealth of possibilities that had previously gone unnoticed. The patience of my students who tolerated my refusal to check my notes or text to see where I had gone wrong is gratefully acknowledged. The thoughtful and continuing encouragement of my colleague, Professor Dale Taulbee, is also much appreciated.

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