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A NEW HEAT TRANSFER LAW FOR TURBULENT NATURAL CONVECTION BETWEEN HEATED HORIZONTAL SURFACES OF LARGE EXTENT

by

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T^+	$(T-T_w)/T_I$
u_o	Outer velocity scale, eq. (19)
w	Fluctuating vertical velocity
z	Vertical coordinate
z^+	z/η_I, inner coordinate
$\tilde{z}$	z/h, outer coordinate
α	Thermal diffusivity
β	Thermal expansion coefficient
ΔT	$T-/T_o$
ΔT_w	Temperature differences between surfaces $2(T_w-T_c)$
η_I	Inner length scale, eq. (11)
ν	Kinematic viscosity
ρ	Density
ρ_o	Reference density
θ	Fluctuating temperature

1. INTRODUCTION

The problem of turbulent natural convection above horizontal surfaces has been a subject of considerable interest over the past 35 years. One reason for this popularity lies in the common occurrence of such flows in engineering and natural environments. Other reasons are because such flows represent an important class of flows in which turbulence is produced only by buoyancy, and they can be easily idealized both analytically and experimentally. Examples of these idealized flows include the unsteady and steady boundary layers above a heated horizontal surface of infinite extent, and the steady state flow between horizontal parallel surfaces of infinite extent.

The first theoretical analysis of turbulent natural convection above a heated horizontal surface was put forth by Malkus (1954) in which he argued for the existence of a z^{-2} region in the mean temperature gradient, where z is the distance from the surface. In the same year, Priestly (1954) on dimensional grounds argued instead for the existence of a region in which the temperature gradient varied as $z^{-4/3}$. Long (1976) and Panofsky (1978) used matched asymptotic analyses to support Priestly's hypothesis.

Another aspect of Long's matching analyses (1976) was the derivation of an asymptotic heat transfer law in which $Nu \sim Ra^{1/3}$ with a Prandtl number dependent coefficient. This is consistent with the empirical heat transfer correlation of Globe and Dropkin (1959) who proposed for the flow between parallel surfaces of large extent,

$$Nu = 0.069\, Pr^{0.074}\, Ra^{1/3}$$

Other investigators (eg. Chu and Goldstein (1973), Fitzjarrald (1976) report values of the Rayleigh number exponent to be somewhat smaller than 1/3, an effect which Long (1976) attributes to the experiments not having reached the asymptotic value.

Adrian et al. (1986) and Fitzjarrald (1976) using their own data argue convincingly for the existence of inner and outer scales, the former proposed by Townsend (1959) and the latter by Deardorff (1970). As to the existence of the power law regions proposed above, Adrian et al. (1986) argue that neither result is particularly satisfactory alone. Instead they provide a qualified endorsement of the meso-layer theory proposed by Chern and Long (1980) in which a z^{-2} layer exists beneath a $z^{-4/3}$ layer in the temperature gradient.

The purpose of this paper is to present a different analysis of the so-called "turbulent Rayleigh problem" using the methodology of George and Capp (1979) for turbulent natural convection next to vertical surfaces. This paper will present only the results for the mean temperature profile and the heat transfer relation. Of special interest is the derivation of the leading correction terms for the heat transfer coefficient at finite Rayleigh number. Some of the features of this analysis have already been proposed, eg. Long 1976. However, it is hoped that this new treatment will provide a more unified approach to the problem of natural convection above horizontal surfaces, and will be particularly useful in understanding the experimental data. The theory is based on an inner-outer scaling approach to the problem, and predicts both the existence of the $z^{-4/3}$ temperature gradient and the asymptotic heat transfer law $Nu \sim Ra^{1/3}$ with a Prandtl number dependent coefficient. An examination of the experimental data from the perspective of this unified theory will be seen to provide strong support for the conclusions.

The remainder of this paper is divided into two parts: Part I will analyze in detail the fully turbulent flow between differentially heated horizontal parallel plates, while Part II will consider the experimental data.

PART I: THE RAYLEIGH PROBLEM

2. The Equations of Motion for Steady Flow

The flow to be analyzed consists of the thermal convection which develops at high Rayleigh number between differentially heated horizontal plates in a gravitational field. It is hypothesized that the flow is fully turbulent and homogeneous in planes perpendicular to the vertical (defined by the gravitational vector). The flow is further assumed to be statistically stationary so that all time derivatives vanish in the averaged equations of motion. The flow and coordinate system is illustrated in Figure 1. The notation utilized throughout will conform as closely as possible to that used by George and Capp (1979).

Since there can be no mean flow in any direction (by hypothesis) the mass conservation equation vanishes identically, and the averaged equations for momentum and temperature to within the Boussinesq approximation reduce to

$$\text{temperature:} \quad 0 = \frac{\partial}{\partial z}\left[-\overline{w\theta} + \alpha\frac{\partial T}{\partial z}\right] \tag{1}$$

$$\text{z-momentum:} \qquad 0 = \frac{\partial}{\partial z}\left[\overline{-w^2} - \frac{P-P_o}{\rho_o}\right] + g\beta(T-T_o) \tag{2}$$

where α is the thermal diffusivity, g is the gravitational acceleration, and β is the thermal expansion coefficient. The reference values denoted by subscript zero can be taken to represent a hypothetical undisturbed state of uniform density. Note that unlike other boundary layer type flows, the pressure variation across the flow can not be neglected by reference to the momentum equation in other directions. Note also that the absence of the viscous term in equation (2) does not represent a high Reynolds number approximation, but rather is a consequence of the absence of a mean flow. Thus viscosity can affect the averaged motion only by its pressure in the higher moment equations.

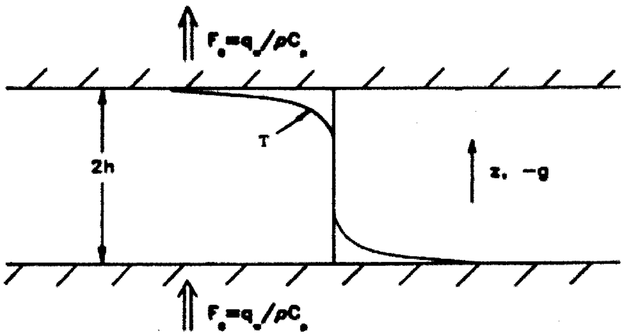


Figure 1. Sketch of Rayleigh flow.

Both equations can be integrated with respect to z to yield

$$\overline{-w\theta} + \alpha \frac{\partial T}{\partial z} = \frac{q_w}{\rho C_p} = -F_o \tag{3}$$

and

$$\left[\overline{w^2} + \frac{P-P_o}{\rho_o}\right] = \int_0^z g\beta\Delta T dz' \tag{4}$$

where q_w is the heat flow at the surface where z=o and must be equal and opposite to that at the other surface for steady flow to exist. Hereafter, for simplicity, the related quantity F_o will be referred to as the heat flux. Equation (3) is of primary importance since it makes it clear that the total heat flux is constant from top to bottom. Thus in the terminology of George and Capp (1979), the entire flow is a <u>constant heat flux "layer"</u>. A consequence of this is that the flow is symmetrical about the centerline and only half of it need be considered.

3. <u>The Existence of an Inner Layer</u>

It is instructive to examine the relative order of magnitude of the terms in the governing equations, especially the temperature equations. It is straightforward to show that in the limit as $2F_oh/\alpha\Delta Tw$ (the Nusselt number) increases (it will be shown below that it is in fact the Rayleigh number which governs the development), the conduction term vanishes over most of the flow and the dynamical equations reduce to

$$F_o = -\overline{w\theta} \tag{5}$$

and

$$0 = \frac{\partial}{\partial z}\left[\overline{-w^2} - \frac{P-P_o}{\rho_o}\right] + g\beta\Delta T \tag{6}$$

Therefore, in this high Nusselt (or Rayleigh) number limit, <u>neither</u> the viscosity nor the thermal conductivity directly affect the averaged momentum or temperature equations.

It is clear that equations (5) and (6) can at most describe a region away from the wall since they can no longer satisfy the surface boundary condition

$$F_o = -\alpha \left.\frac{\partial T}{\partial z}\right|_{z=o} \tag{7}$$

Thus, in like manner to the analysis of George and Capp (1979), equations (5) and (6) must be recognized as governing an <u>outer</u> flow region away from the vicinity of the walls. It follows immediately that there must exist a region near to the walls in which a different set of equations applies which retains both the conduction and the turbulence heat transfer terms. An appropriate scaling length for this wall region must be sought which allows the conduction term to be retained there. If this inner length scale is denoted as η_I, then η_I must be chosen so that the conduction term is at least as important as the turbulence term. Thus in this inner layer the governing equations are exactly equations (3) and (4).

The inner layer can be further subdivided by noting that the kinematic boundary condition ($\underline{u}\cdot\underline{n}=0$ at a fixed surface) causes $\overline{w\theta}$ to vanish identically at z=0. Therefore very close to the wall, the temperature equation reduces to

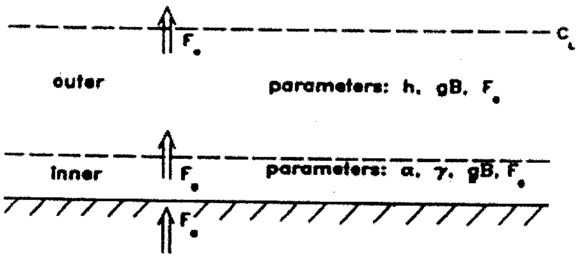
$$\alpha \frac{\partial T}{\partial z} = -F_o \tag{8}$$

This can be immediately integrated to yield

$$T - T_w = -\frac{F_o}{\alpha} z \tag{9}$$

so that this inner-inner region can be recognized as a layer in which the mean temperature varies linearly. It is appropriate (again following George and Capp (1979)) to refer to this sub-region as the <u>Conductive Sublayer</u>.

An examination of the dynamical equations in the limit as $2F_oh/\alpha\Delta T_w \to \infty$ has led to the recognition that the natural convection flow between parallel horizontal surfaces can be characterized by two layers: an inner layer in which the conduction term allows the surface boundary condition to be met, and an outer layer where conduction effects are negligible. The inner layer has been shown to include a region very close to the wall in which the mean temperature varies linearly, the <u>conductive sublayer</u>. A schematic of the flow is shown in Figure 2 along with and parameters governing each region. Note that the kinematic viscosity has been included for the inner layer, even though it does not directly enter the first-order moment equations but enters only in the higher order equations for the turbulence heat flux and the turbulence kinetic energy.



igure 2. Schematic showing inner and outer layers.

. ΔT_w versus F_o

In order to establish scaling laws for the inner ıd outer regions identified above, it is necessary to ·cide which of the parameters arising in the ıuations of motion and boundary conditions govern ıch region. It is easy to see that it is improper to ;e the temperature difference between the two ırfaces, ΔT_w, to scale the temperature variation in ither layer since only part of the total temperature :op occurs across each. Thus a proper scaling ıalysis must include new measures of the temperature :ales for the inner and outer regions.

The key to properly scaling the inner and outer ıyers lies in the recognition that the entire flow is ıaracterized by a constant heat flux (total heat flux ıdependent of height). The inner layer has the ırface heat flux, F_o, imposed on it. Because of the ınstancy of the total heat flux across the inner ıyer, F_o also provides an inner boundary condition on ıe outer (or core) flow. Thus F_o is a governing ırameter for both the inner and outer regions. The ·maining parameters must be obtained from an :amination of the governing equations and boundary ınditions.

The Inner Layer

From the equations of motion for the inner layer ·quations 2.3 and 2.4) it is clear that the only ırameters arising in the equations are $\alpha, g\beta$, and ν. ıus the inner layer must be governing entirely by the ırameters F_o, $g\beta$, α and ν. An inner temperature :ale can be defined as

$$T_I = \frac{F_o^{3/4}}{(g\beta\alpha)^{1/4}} \tag{10}$$

ıd an inner length scale as

$$\eta_I = \left[\frac{\alpha^3}{g\beta F_o}\right]^{1/4} \tag{11}$$

Note that these choices are not unique since ν ·uld have been used in place of α. In fact, any ·mbination of α and ν having the correct dimensions ·uld have been used. It is this ambiguity which ·quires that the Prandtl number dependence be ·tained in the scaled profiles given below for the ner layer. The particular choices made here were rst introduced for this problem by Townsend (1959) d are exactly those used by George and Capp (1979).

The mean temperature profile in inner variables can now be written as

$$T - T_w = T_I \; F_I(z^+, \; Pr, \; H_*) \tag{12}$$

where z^+ is a dimensionless inner coordinate defined by

$$z^+ = z/\eta_I \tag{13}$$

$Pr = \nu/\alpha$ is the Prandtl number and H_* is the flux H-number (v. George and Capp (1979) defined as

$$H_* = \frac{g\beta F_o h^4}{\alpha^3} \tag{14}$$

It is important to note that the temperature has been referenced to its value at the wall in order to avoid the need to directly account for effects from outside the wall region.

The flux H-number takes on additional significance as having a direct relation to the ratio of gap half-width, h, to the inner length scale, η_I. From the definitions of equations (11) and (14) it follows that

$$\frac{h}{\eta_I} = H_*^{1/4} \tag{15}$$

This is particularly important since h will be identified below as the outer length scale. The inner-outer nature of the problem as $H_* \to \infty$ is again clear.

It is important to note that equation (12) describes not only the temperature variation in the inner region, but across the entire flow, the dependence on H_* reflecting the dependence of the inner flow on the outer at finite values of H_*. In the limit as $H_* \to \infty$, the dependence on this variable must vanish (if the inner variables have been correctly chosen), and equation (12) will describe only the inner temperature profile or a Law of the Wall (Adrian et al.(1986). George and Capp (1979)).

6. The Conductive Sublayer

The linear profile of equation (9) can be rewritten in the inner variables proposed above. The result is

$$\frac{T - T_w}{T_I} = -\frac{z}{\eta_I} \tag{16}$$

or

$$F_I(z^+) = -z^+ \tag{17}$$

This particular functional form describes only the conductive sublayer portion of the inner layer where the turbulence heat flux contribution can be neglected. The exact extent of this region will depend on the Prandtl number, since viscous effects control the emergence of $\overline{w\theta}$ as the distance from the wall increases. If the Prandtl number is large, disturbances will be damped to a greater distance from the wall than the distance for which the temperature equation can be expected to be described by the linear term only, As a consequence, the temperature will begin to show a gradual deviation from the linear dropoff between distances characterized by η_I and $Pr^{3/4}\eta_I$, the latter characterizing the extent of the viscous-dominated region. On the other hand if the Prandtl number is low, the linear dropoff in temperature can be expected to continue for distances well beyond the extent of the viscous region.

7. The Outer (or Core) Region

From the equations of motion (equations (5) and (6), it is clear that the only parameters governing the outer flow are $g\beta$ and F_o, the former occurring directly in the governing equations, and the latter from the boundary condition imposed by the inner layer. Since the inner layer becomes vanishingly small as the H-number increases, the appropriate length scale for the outer flow is the half-distance between the surfaces, h. From these parameters, outer temperature and velocity scales can be uniquely defined as

$$T_o = F_o^{2/3}/(g\beta h)^{1/3} \tag{18}$$

and

$$u_o = (g\beta F_o h)^{1/3} \tag{19}$$

These scales correspond directly to these utilized by George and Capp (1979) for vertical surfaces, and were first introduced into this problem by Deardorff (1970).

The variation of temperature in outer variables can now be written as

$$T - T_c = T_o\ f_1(\tilde{z},\ H_*,\ Pr^{-3}H_*) \tag{20}$$

where $\tilde{z}$ is a dimensionless outer coordinate defined by

$$\tilde{z} = z/h \tag{21}$$

Here the dependence on H_* and $Pr^{-3}H_*$ reflects the influence of the inner parameters at finite values of H_* and $Pr^{-3}H_*$.

T_c, the centerline temperature, is chosen as a reference temperature to avoid referencing the temperature variation to a value outside the core region. <u>This is necessary to avoid the need to include explicitly the conduction and viscous effects which influence the variation of temperature across the inner layer.</u> (They are, however, included implicitly through H_*.) Thus the term "Deficit Laws" is appropriate to this outer form. Note that similar considerations must also apply to other statistical quantities (which are not considered here). While the earlier analyses have been generally careful about properly referencing the mean temperature, the need to properly reference the other quantities appears to have been overlooked until now.

In the limit as $Pr^{-3}H_*$, $H_* \to \infty$, the function $f_1(\tilde{z}, Pr^{-3}, H_*)$, must become asymptotically independent of both. This is because it represents a profile non-dimensionalized in outer variables and must remain finite in the outer region. Thus in this limit, equation (20) represents a true outer profile, having lost all dependence on viscous and conduction effects.

8. A Matched Layer: The Buoyant Sublayer

It has been noted above that the functional forms of equations (12) and (20) reduce to inner and outer profiles respectively in the limit as $Pr^{-3}H_*, H_* \to \infty$. It is interesting to ask whether there exists a range of distances for which both the inner and outer scaling laws are valid - in effect, an overlap range. The procedure is exactly equivalent to that followed by George and Capp (1979) and requires matching the inner and outer profiles in the limit of infinite flux H-number. In formal terms: Is there a region of common validity in the limit as Pr^{-3} H_*, $H_* \to \infty$ so that

$$\lim_{\substack{z/h \to 0 \\ H_* \to \infty \\ Pr^{-3}H_* \to \infty}} f_1\left[\frac{z}{h}, H_*, Pr^{-3}H_*\right] = \lim_{\substack{z/\eta_I \to \infty \\ H_* \to \infty \\ Pr^{-3}H_* \to \infty}} F_1\left[\frac{z}{\eta_I}, Pr, H_*\right] \tag{22}$$

It is more convenient at this point to match the temperature derivatives given in inner and outer variables as

$$\frac{\partial T}{\partial z} = \frac{T_I}{\eta_I} F_1'(z^+, Pr, H_*) \tag{23}$$

and

$$\frac{\partial T}{\partial z} = \frac{T_o}{h} f_1'(\tilde{z}, H_*,\ Pr^{-3}H_*) \tag{24}$$

where ′ denotes differentiation with respect to the appropriate variable. Equating these in the limit as $Pr^{-3}H_*$, $H_* \to \infty$ yields

$$\frac{T_I}{\eta_I} F_1'(z^+, Pr) = \frac{T_o}{h} f_1'(\tilde{z}) \tag{25}$$

From equations (10) and (18) it follows that

$$\frac{T_o}{T_I} = \left[\frac{\eta_I}{h}\right]^{1/3} = H_*^{1/12} \tag{26}$$

Using this and multiplying both sides of eq. (25) by $z^{4/3}$ yields

$$(z^+)^{4/3}\ F_1'(z^+, Pr) = \tilde{z}^{4/3} f_1'(\tilde{z}) \tag{27}$$

Since $h/\eta_I \to \infty$ as $H_* \to \infty$, it follows from the definitions of z^+ and $\tilde{z}$ that their ratio becomes undefined in this limit. Thus z^+ and $\tilde{z}$ become independent variables as $H_* \to \infty$. As a consequence, both sides of equation (27) must equal a constant, say $-K_1/3$. Thus,

$$(z^+)^{4/3}\ F_1'(z^+, P_r) = -K_1/3 \tag{28}$$

and

$$(\tilde{z})^{4/3}\ f_1'(\tilde{z}) = -K_1/3 \tag{29}$$

Equations (28) and (29) can be readily integrated to yield

$$F_1(z^+, Pr) = K_1(z^+)^{-1/3} + B_1(Pr) \tag{30}$$

$$f_1(\tilde{z}) = K_1(\tilde{z})^{-1/3} + B_{1o} \tag{31}$$

where K_1 and B_{1o} are universal constants and $B_1(Pr)$ is a universal function of the Prandtl number.

Equations (31) and (32) can be expressed in physical variables as

$$\text{inner} \qquad \frac{T - T_w}{T_I} = K_1\left[\frac{z}{\eta_I}\right]^{-1/3} + B_1(Pr) \tag{32}$$

outer $$\frac{T-T_c}{T_o} = K_1\left[\frac{z}{h}\right]^{-1/3} + B_{1o} \qquad (33)$$

Equations (32) and (33) thus represent the inner and outer forms of the temperature profile in a matched region between the inner and outer layers. This layer (by analogy with the analysis by George and Capp 1979) will be referred to as the Buoyant Sublayer, since it can be shown to be governed by only the distance to the wall and the buoyancy parameter.

The existence of such a region has been previously derived on the following dimensional grounds by Priestley (1954), (see also Turner (1973). Suppose there exists a region for which η_I and $Pr^{3/4}\eta_I$ << z << h. This can, of course, occur only in the limit of large H_*. Since the region of interest is well-removed from viscous or conductive effects, yet too small to be dependent on gap width, it must be entirely determined by $g\beta$, F_o, and z itself. Thus, on dimensional grounds (considering $g\beta$ to be a single parameter since they always occur in the equations in combination),

$$\frac{dT}{dz}\,\frac{(g\beta)^{1/3}z^{4/3}}{F_o^{\,2/3}} = \text{constant} = -\frac{K_1}{3} \qquad (34)$$

This can readily be integrated to yield equations (32) and (33). The origin of the Prandtl number dependence of the constant in equation (32) can be seen to arise from the reference to the wall temperature and the integration over the viscous region.

Although only the temperature profile is being considered here, it is noted that a matching of inner and outer profiles can also be carried out for $\overline{w^2}$, $(P-P_o)/\rho_o$, or any other turbulence quantity, and an appropriate buoyant sublayer form deduced. As for the mean temperature gradient, the functional dependences of $d\overline{w^2}/dz$ and dP/dz could have been deduced on dimensional grounds as noticed by Adrian et al. (1986). It is important to note, however, that because of the need to exclude the dependence on α, ν, and h, dimensional arguments for the buoyant sublayer can only be applied to the variation of T, $\overline{w^2}$ and P within the sublayer, and not to these quantities directly. Thus, it is appropriate to argue that $dT/dz \sim z^{-4/3}$ but not that $T \sim z^{-1/3}$; similarly that $\partial\overline{w^2}/\partial z \sim z^{-1/3}$ but not $\overline{w^2} \sim z^{2/3}$, etc. The matching results above make clear the importance of both the additive constants and the use of appropriate reference values. The failure to recognize this clearly before now is largely responsible for the frustration of experimenters who sought to confirm the existence of the buoyant sublayer by using log-log plots of T and $\overline{w^2}$ versus z (see for example Adrian et al.(1986), Goldstein and Chu (1969)).

9. A Heat Transfer Law

Since both inner and outer forms of the dependent variables in the buoyant sublayer describe the same physical profiles, the actual (unscaled) variables predicted by both must be the same. For example, equations (32) and (33) must yield the same temperature at the same location. Thus

$$T_w + T_I\left\{K_1\left[\frac{z}{\eta_I}\right]^{-1/3} + B_1(Pr)\right\}$$

$$= T_c + T_o\left\{K_1\left[\frac{z}{h}\right]^{-1/3} + B_{1o}\right\} \qquad (3$$

Using the definitions of T_I, η_I and T_o (equatio (10), (11) and (18) it follows after rearranging tha

$$\frac{T_w-T_c}{T_I} = \left[\frac{\eta_I}{h}\right]^{1/3} B_{1o} - B_1(Pr) \qquad (3$$

This can be transformed into a more familiar fo by using a Nusselt number defined as

$$Nu = \frac{F_o h}{\alpha(T_w-T_c)} \qquad (3$$

Note that because of symmetry, the temperatu difference between surfaces is

$$\Delta T_w = 2(T_w-T_c) \qquad (3$$

Thus, since the distance between the plates is 2h, t Nusselt number of equation (38) is just the usu Nusselt number based on temperature difference and g width.

By using equations (10), (11), (15) and (37 equation (36) can be rewritten as

$$Nu^{-1}\,H_*^{1/4} = B_{1o}\,H_*^{-1/12} - B_1(Pr) \qquad (3$$

or

$$Nu = \frac{H_*^{1/4}}{B_{1o}H_*^{-1/12} - B_1(Pr)} \qquad (4$$

Thus the heat transfer law is completely determined t second order by the buoyant sublayer constants of t mean temperature profile. This appears not to ha been noticed before now, and provides an importa check on the consistency of experimental data as we as on the internal consistency of the theory.

In the limit as $H_* \to \infty$,

$$Nu = C_H'\,H_*^{1/4} \qquad (41$$

where

$$C_H' = [-B_1(Pr)]^{-1} \qquad (42$$

Thus at very large values of H_*, the Nusselt numbe varies as $H_*^{1/4}$ with a coefficient dependent o Prandtl number. However, for smaller values of as H (yet still large enough to ensure the validity of th theory), the $H_*^{-1/12}$ correction term will modify th coefficient. It will be seen later that both B_1(Pr and B_{1o} are negative so that the effect of finite H is to increase the Nusselt number above the valu which would be predicted by equation (41) alone. Not that the negative value of B_{1o} implies that the theor breaks down long before the $H_*^{-1/12}$ term become dominant in the denominator; thus an $H_*^{1/3}$ asymptot will not be observed at moderate values of H_*.

It has been customary in the experimenta literature to correlate data using the Rayleigh numbe and the Nusselt number. The Rayleigh number based o h and (T_w-T_c) can be defined as

$$Ra = \frac{g\beta(T_w - T_c)h^3}{\alpha\nu} \tag{43}$$

which can be related to the corresponding H- number by

$$Ra = H \cdot Pr \tag{44}$$

where

$$H = \frac{g\beta(T_w T_c)h^3}{\alpha^2} \tag{45}$$

(Note that the usual definitions are based on the temperature difference and distance between the surfaces and are therefore $2^4=16$ times bigger than the H and Ra defined here.)

From the definitions of equations (14), (37), and (45), it follows that

$$H_* = H \cdot Nu \tag{46}$$

Thus the heat transfer law of equation (40) can be transformed into

$$Nu = \frac{H^{1/4} Nu^{1/4}}{\left[B_{10} H_*^{-1/12} Nu^{-1/12} - B_1(Pr)\right]} \tag{47}$$

or

$$Nu = \frac{H^{1/3}}{\left[B_{10} H^{-1/12} Nu^{-1/12} - B_1(Pr)\right]^{4/3}} \tag{48}$$

The transformed heat transfer law of equation (48) is considerably more complicated than equation (40) because of the implicit dependence on the Nusselt number*. Nonetheless, it is easy to see that for very large H- numbers, the Nusselt number varies as the cube root of the H- number and the coefficient is Prandtl number dependent, i.e..

$$Nu = C_H(Pr)\, H^{1/3} = C_H(Pr) Pr^{1/3} Ra^{1/3} \tag{49}$$

where

$$C_H(Pr) = \left[\frac{-1}{B(Pr)}\right]^{4/3} \tag{50}$$

The Prandtl number dependence is not determined by the analysis (to this point, at least) and must be derived from other considerations.

The recognition that $Nu \sim Ra^{1/3}$ with a Prandtl number-dependent coefficient could be derived by an asymptotic matching analysis was apparently first noticed by Long (1975). His arguments were somewhat different, however, and led to two different dependence for smaller values of the Rayleigh number than that proposed here one of which is equivalent to equation (47).. The correction for finite values of H_* given by equation (47) will act to increase the heat transfer coefficient so that the experimentally determined dependence on Rayleigh number will be <u>less</u> than the 1/3 power. This will be discussed further in Part II when experimental data are considered.

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*This is not a serious limitation in practice because the Nusselt number can first be calculated for given H_* using equation (40), then the value of H assigned using equation (46).

It will be shown in a subsequent publication that a similar matching can be carried for other statistical quantities which enter the governing equations (eg. $\overline{w^2}$, $\overline{\theta^2}$, etc.). The results of matching the appropriate inner and outer buoyant sublayer forms are "laws" which describe the variation of the centerline values of these quantities with H_*. These possibilities do not appear to have been noticed previously.

PART II: EVALUATION OF THEORY USING EXPERIMENTAL DATA

10. <u>Overview of Experiments</u>

There have been a number of laboratory experiments over the past two decades which attempted to simulate the fully developed turbulent Rayleigh problem. There have also been experimental investigations of several related flows including the natural convection between parallel plates with an adiabatic upper surface (Adrian et al. 1986), steady penetrative convection (Townsend 1959), unsteady penetrative convection (Deardorff et al. (1969). In addition there have been numerous investigations of the convective planetary boundary layer (Wyngaard et al. 1971, see also Monin and Yaglom, vol. I 1971). While there is reason to believe that all these flows should obey common scaling laws, there is little reason to believe that the scaled profiles should be the same for all, except perhaps in the inner layer. For now this question will be avoided, and only the Rayleigh experiments will be considered below.

11. <u>The Mean Temperature Profiles: Inner Variables</u>

There have been several experiments carried out which measured the mean temperature distribution at reasonably high Rayleigh numbers (eg. Somerscales and Gazda 1969, Chu and Goldstein 1973, Goldstein and Chu 1969, and Deardorff and Willis 1967a, 1967b). Many of these investigators made a particular effort to evaluate whether or not there was a region in the temperature gradient which varied as $z^{-4/3}$ (or the temperature varied as $z^{-1/3}$). Of these, only Deardorff and Willis (1967b) were partially successful in identifying a limited $z^{-4/3}$ region in the temperature gradient. It is suggested here that the failures were due in part to the relatively low values of the Rayleigh number which limited the extent of the buoyant sublayer, but in larger part to the use of log-log plots in the presence of additive constants, and the failure to understand clearly where the proposed region should lie.

Figure 3 is a plot of the Deardorff and Willis data (1967b) in inner variables. The data were read from enlarged versions of the figures in their paper, and the scatter in the fitted curves is largely due to the errors in this process, especially near the center of the flow. In order to make readily apparent the existence of a region described by equation (32), the data are plotted as $[(T-T_w)/T_1][z/\eta_1]^{1/3}$ versus $[z/\eta_1]^{1/3}$. Thus the buoyant sublayer, should it exist at all, would correspond to a straight line. Note also that since the plot is in inner variables, the data should be expected to deviate at progressively larger values of z^+ as the Rayleigh number of the experiment is increased since the ratio of outer to inner scales increases.

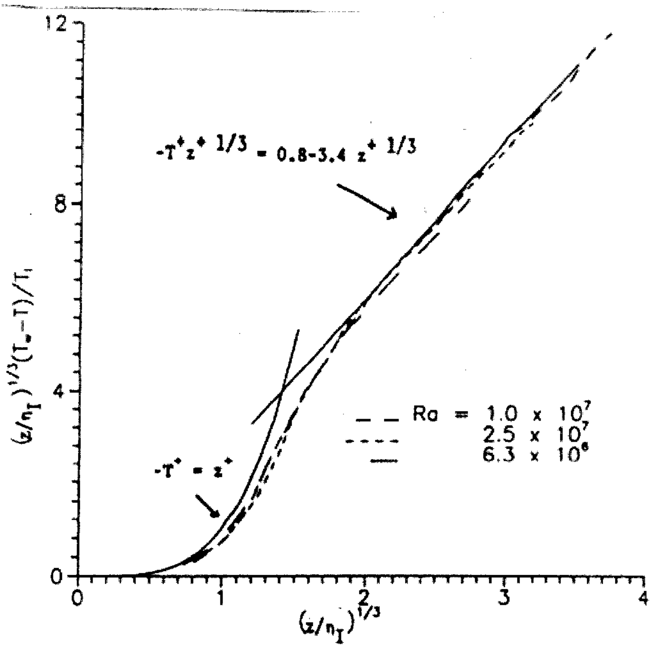

Figure 3. Mean temperature versus height in inner variables, air data of Deardorff & Willis.

It is clear from Figure 3 that the data collapse well in inner variables until very large values of z^+ are obtained, and that the value of z^+ for which the curves begin to deviate from each other increases with Ra as expected. The region nearest the wall ($z^+<1$) is well-described by the linear dependence of equation (17). Of greater interest is the existence of the $z^{-1/3}$ region as represented by the straight line on the plot. This method of plotting allows determination of both K_1 and $B_1(Pr)$ directly, and also makes clear that the buoyant sublayer (or $z^{-1/3}$ region) is an asymptotic region which will be increasingly apparent with increasing Ra (or H_*).

In Figure 4, the same data are plotted following George and Capp (1979) as $(T-T_w)/T_I$ versus $(z/\eta_I)^{-1/3}$. This method would be of considerable value if profiles from different Prandtl number fluids were being plotted together. According to equation (32), the effect of Prandtl number should be to shift the $z^{-1/3}$ region, but to leave its slope unchanged. This has not been confirmed for this problem since only the air data have been considered to-date.

The constants K_1 and $B_1(Pr)$ of equation (32) have been estimated from a least squares fit to the highest Ra number experiment using data from $8 < z^+ < 27$ ($2 < z^{+1/3} < 3$). The results are

$$K_1 = 0.73 \tag{51}$$

$$B_1(0.71) = -3.4 \tag{52}$$

The constant K_1 is related to the C_1 constant of Deardorff and Willis (1967b) by $K_1 = C_1/3$. They found by plotting dT/dz versus z that $C_1 = 2.2$ for the same data which corresponds exactly to the value obtained here. The constant $B_1(0.71)$ is, of course, valid only for air ($Pr = 0.71$) and a different value can be expected for a different fluid. Note that B(Pr) enters directly the heat transfer laws (equations (40) and (50)), and so this value will be considered again below.

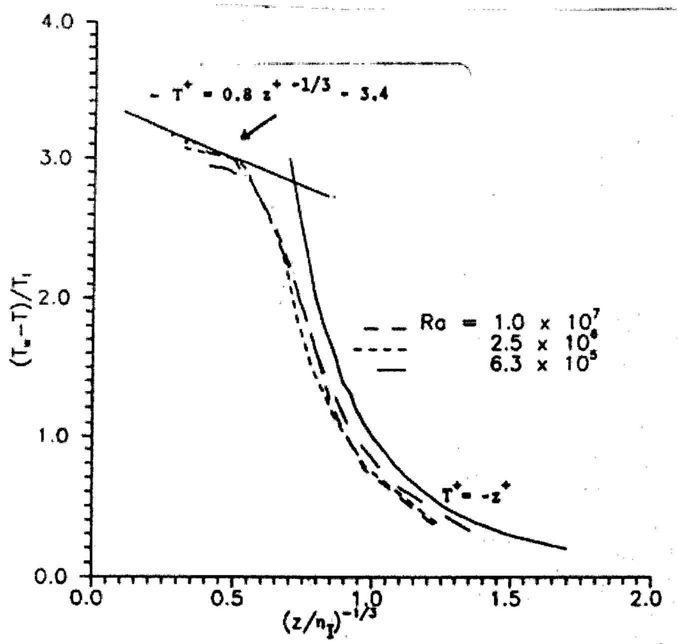

Figure 4. Mean temperature versus height in inner variables, air data of Deardorff and Willis.

12. The Mean Temperature Profile: Outer Variables

The same mean temperature profile data as that used above has been replotted in outer variables in Figure 5. Considerable difficulty was encountered in trying to plot the data in outer variables to make clear the dependence over most of the outer flow. This was because most of the temperature drop occurred in the inner layer (in fact, the linear layer) so that the temperature outside the inner region was very close to the centerline value to which it must be referenced. Thus $(T-T_c)$ was very sensitive to the errors in determining both T and T_c. In view of the lack of alternatives, the data were plotted as $(T-T_c)/T_s$ versus $(z/h)^{-1/3}$. Thus the left ordinate is $(z/h)^{-1/3} = 1.0$ and corresponds to the centerline; increasing values of $(z/h)^{-1/3}$ are progressively closer to the wall. Again the buoyant sublayer (or $z^{-1/3}$ region) should correspond to a straight line.

As expected, the outer scaling works reasonably well for value of $\tilde{z}=z/h$ greater than about $0.1(\tilde{z}^{-1/3}<2)$. The deviations at smaller values of $\tilde{z}$(larger $\tilde{z}^{-1/3}$) are a consequence of the fact that the region nearest the wall collapses only in inner variables. In spite of the scatter and the low values, there does appear to be a straight line region on the plot, especially at the higher Rayleigh numbers. From it the constants K_1 and B_{10} can be determined. Because of the large errors, K_1 was taken at the value determined above, and the value of B_{10} was determined by inspection to be

$$B_{10} = -1.0 \tag{53}$$

Obviously there is further work needed to refine this estimate. Note that whatever the value is, it should be universal and independent of Prandtl number.

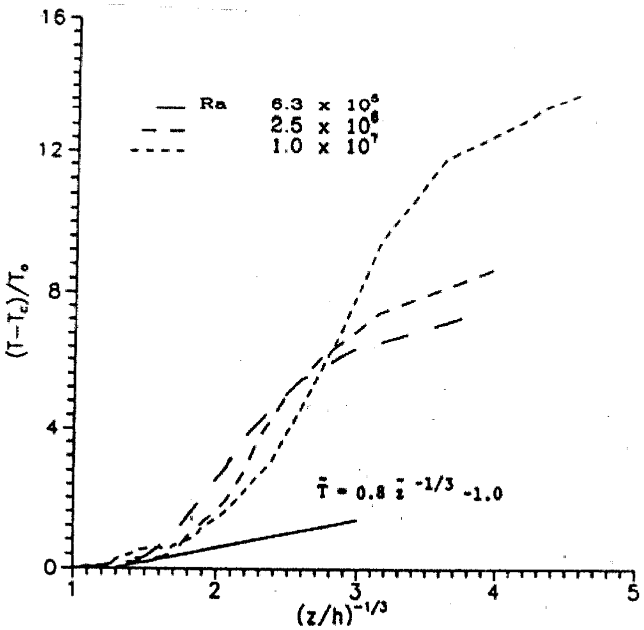

Figure 5. Mean temperature difference from centerline value versus height in outer variables, air data of Deardorff & Willis.

13. The Heat Transfer Law

The relation between Nusselt number and Rayleigh number has been the subject of intense investigation with almost as many proposed laws as investigators. Several of the suggested empirical relations for air are given below

Fitzjarrald (1976) (air, $8 \times 10^4 < Ra < 9 \times 10^9$)

$$Nu = 0.13\ Ra_d^{0.30} \tag{54}$$

Goldstein and Chu (1969)(air $3 \times 10^5 < Ra < 10^8$)

$$Nu = 0.123\ Ra_d^{0.294} \tag{55}$$

Globe and Dropkin (1959)

$$Nu = 0.0673\ Ra_d^{1/3} \tag{56}$$

where Ra_d is 16 times the Ra used earlier. Deardorff and Willis (1967a) note that their experiments in air are not too different from equation (56).

It was shown in Section 9 that the matching of the inner and outer temperature profiles yielded a second order heat transfer law (equation 40) with the coefficients determined only by the matching. Thus for air using $B_1(0.71) = -3.4$ and $B_{10} = -1.0$ in equation (40) yields

$$Nu = \frac{0.294\ H_*^{1/4}}{\left[1 - 0.294\ H_*^{-1/12}\right]} \tag{57}$$

This can be evaluated for given H_*, then the appropriate Rayleigh number calculated from equations (44), (45) and (46). The results are shown in Table I and Figure 6. Also shown are equations (54) and the asymptotic law ($H_* \to \infty$) determined from equation (57) as

$$Nu = 0.294\ H_*^{1/4} \tag{58}$$

or

$$Nu = .0690\ Ra_d^{1/3} \tag{59}$$

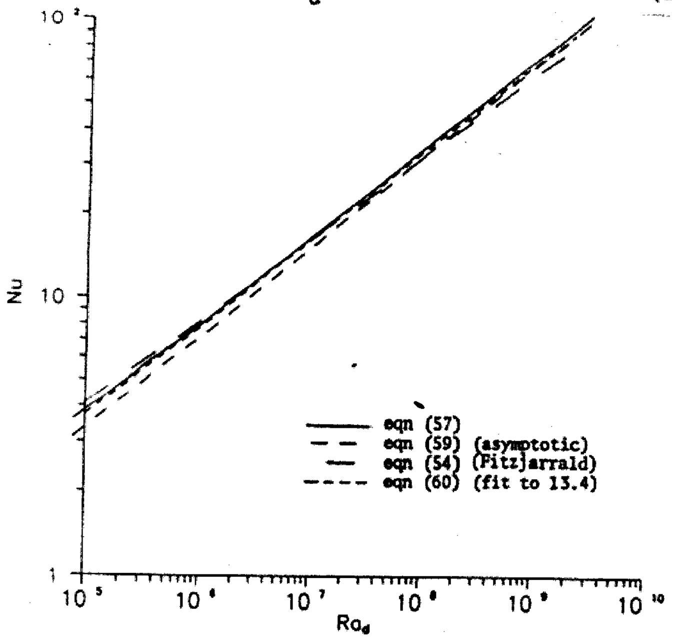

Figure 6. Comparison of theoretical and empirical heat transfer laws.

Table 1

Tabulation of Nusselt Number for Air Using Equations (40) and (41).
$B_1(0.71) = -3.4$, $B_{10} = -1.0$

H_*	Nu	Nuasym	Ra_d
1.00E+04	3.41E+00	2.94E+00	6.62E+04
3.00E+04	4.42E+00	3.87E+00	1.53E+05
1.00E+05	5.89E+00	5.23E+00	3.82E+05
3.00E+05	7.67E+00	6.88E+00	8.81E+05
1.00E+06	1.03E+01	9.30E+00	2.20E+06
3.00E+06	1.34E+01	1.22E+01	5.05E+06
1.00E+07	1.79E+01	1.65E+01	1.26E+07
3.00E+07	2.34E+01	2.18E+01	2.89E+07
1.00E+08	3.14E+01	2.94E+01	7.18E+07
3.00E+08	4.11E+01	3.87E+01	1.65E+08
1.00E+09	5.52E+01	5.23E+01	4.08E+08
3.00E+09	7.23E+01	6.88E+01	9.35E+08
1.00E+10	9.72E+01	9.30E+01	2.32E+09
3.00E+10	1.27E+02	1.22E+02	5.31E+09
1.00E+11	1.72E+02	1.65E+02	1.31E+10
3.00E+11	2.25E+02	2.18E+02	3.00E+10
1.00E+12	3.03E+02	2.94E+02	7.44E+10
3.00E+12	3.98E+02	3.87E+02	1.70E+11
1.00E+13	5.36E+02	5.23E+02	4.20E+11
3.00E+13	7.04E+02	6.88E+02	9.60E+11
1.00E+14	9.49E+02	9.30E+02	2.37E+12
3.00E+14	1.25E+03	1.22E+03	5.42E+12
1.00E+15	1.68E+03	1.65E+03	1.34E+13
3.00E+15	2.21E+03	2.18E+03	3.06E+13
1.00E+16	2.98E+03	2.94E+03	7.56E+13

It is clear from the figures that regardless of the exact value of the constant B_{10}, its effect is to increase the heat transfer above that which would be obtained from the cube root relationship. It is also easy to see that this effect will reduce the effective exponent of the Rayleigh number below 1/3, and that the amount of the reduction will depend on the Rayleigh number range considered. For example, a least squares fit to equation (57) in the range $10^5 < Ra < 10^{10}$ yields

$$Nu = .094\ Ra_d^{0.32} \tag{60}$$

The three empirical laws (equations 54, 56 and 60) differ from that given by equation (57) by less than 4% at $Ra = 1 \times 10^7$ and only by 11% at $Ra = 1 \times 10^5$. The agreement could be further improved by selecting a slightly higher value for the outer constant B_{1o} (like −2.3) which may be warranted in view of the difficulty in determining it from the outer profile. Thus over the range in which most experiments were performed, the heat transfer law obtained from matching the profiles is in excellent agreement with the earlier empirical correlations. Equation (40) (and 57) must therefore be regarded as an improved heat transfer law valid for at least $Ra > 10^5$.

No effort has been made to determine theoretically or empirically the Prandtl number dependence of the integration constants for the inner buoyant sublayer profiles (eg. B_1 (Pr), etc.). From the correlation of Globe and Dropkin (1959) it can be inferred that

$$B_1(Pr) = 3.06\ Pr^{-0.305} \tag{61}$$

It follows that the corresponding heat transfer law should be

$$Nu = \frac{0.327\ Pr^{0.305} H_*^{1/4}}{\left[1 - .294\ H_*^{-1/12}\right]} \tag{62}$$

These must be confirmed by further investigation.

14. Summary and Conclusions

A scaling analysis of fully developed turbulent natural convection between differentially heated surfaces of large extent has led to the identification of inner and outer layers. Only the inner layer is affected by the viscosity and thermal diffusivity. Matching of the mean temperature profiles in the limit as $H_* \to \infty$ and $Pr^{-3} H_* \to \infty$ resulted in the identification of a buoyant sublayer in which the mean temperature varied as $z^{-1/3}$ with an additive constant. Further matching of these profiles yielded a heat transfer law valid at finite values of H_*, as well as in the limit.

The available experimental data for air were analyzed in detail. The data were in excellent agreement with the predictions. Especially gratifying was the success of the temperature profile buoyant sublayer constants in producing a heat transfer law which was able to reproduce the experimental observations over the entire range of Rayleigh numbers measured.

It is likely that the results of the analysis presented here are directly applicable to related natural convection problems. The asymptotic value of the ratio $(T_w - T_c)/T_I$ of 3.4 is very close to its counterpart in both steady and unsteady non-penetrative convection (Adrian et al.(1986). Also, the temperature profiles obtained by Townsend (1959) when normalized in inner variables are indistinguishable from those in Figure 3. Thus when a proper accounting is taken of the Prandtl number dependence of the inner layer, it is likely that its characteristics will prove to be flow-independent. It is unlikely that this will be the case for the outer flow, however, (except for the buoyant sublayer part of it) since it will be more directly influenced by the differing boundary conditions away from the surface.

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