SELF-PRESERVATION OF TEMPERATURE FLUCTUATIONS IN ISOTROPIC TURBULENCE

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ABSTRACT

The averaged spectral equations governing the decay of temperature fluctuations in an isotropic turbulent field are considered and found to admit to fully self-preserving solutions which retain a dependence on the initial conditions. The characteristic length scale for the decay is shown to be the scalar Taylor microscale, and the characteristic temperature is the square root of the scalar variance. In addition to recovering the well-known empirically established power law decay for the variance, the mechanical/temperature time scale ratio is shown to be constant. The invariants of scalar decay are discussed and a number of questions are raised which require further research.

1. Introduction

The decay of temperature fluctuations in isotropic turbulence has been a subject of considerable interest since the early 1950's. Corrsin (1951a,b) was the first to write equations describing the dynamics of the spectrum and correlation functions in isotropic flow. Monin and Yaglom (1975) provide a comprehensive review of attempts to scale and close these equations.

There have been numerous attempts over the years to establish the characteristics of the decay of temperature fluctuations introduced into grid-generated turbulence (Mills et al. 1958, Lanza and Schwarz 1966, Yeh and Van Atta 1973, Lin and Lin 1973, Warhaft and Lumley 1978). Warhaft and Lumley

(1978) showed that each attempt generated a decay rate and spectral shape uniquely determined by its initial conditions. In addition they showed that the mechanical/thermal time scale ratio defined by

$$r = \tau/\tau_\theta = (\overline{q^2}/\epsilon)/(\overline{\theta^2}/\epsilon_\theta) \qquad (1)$$

was unlikely to achieve the expected equilibrium value of unity, but were unable to establish a physical reason as to why.

Recently George (1987a,b) has shown that the dynamics of isothermal decaying isotropic turbulence achieve a self-preserving state, and that the spectrum throughout decay can be scaled by a single length scale, the Taylor microscale defined by

$$\lambda^2 = 15\nu\, u^2/\epsilon \qquad (2)$$

and a velocity scale defined from the kinetic energy by

$$u = \left[\frac{1}{3}\,\overline{q^2}\right]^{1/2} \qquad (3)$$

A further consequence of the theory is that the turbulence energy undergoes a power law decay $u^2 \sim t^n$. Both n and the spectral shape are determined by the initial conditions.

It is the purpose of this paper to extend these self-preservation arguments to the scalar field. The result will be shown to again be that the spectrum (this time the scalar spectrum) scales with a single length scale, the thermal Taylor microscale λ_θ, and the scalar intensity, $\theta = (\overline{\theta^2})^{1/2}$. The consequences of self-preservation on other turbulence properties of interest will be explored. Finally, the results of the analysis will be shown to be reasonably consistent with experiment.

2. The Scalar Spectral Equation

The equation governing the evolution of a homogeneous, isotropic passive temperature field, $\theta(\underline{x},t)$, in a homogeneous, isotropic turbulence is given by (Monin and Yaglom 1975)

$$\frac{\partial}{\partial t} E_\theta = T_\theta - 2\,\alpha\, k^2 E_\theta \qquad (4)$$

where α is the thermal diffusivity, $E_\theta = E_\theta\ (k,t)$ is the three-dimensional scalar spectral function (hereafter referred to as the scalar spectrum), and $T_\theta = T_\theta\ (k,t)$ is the scalar spectral energy transfer function. The integral of the scalar spectrum over all wavenumbers is one-half of the scalar variance, i.e.

$$\frac{1}{2}\ \theta^2 = \frac{1}{2}\ \overline{\theta^2} = \int_0^\infty E_\theta\ (k,t)dk \tag{5}$$

It can also be shown (Monin and Yaglom 1975) that the scalar dissipation rate, ϵ_θ, is given by

$$\epsilon_\theta = -\frac{d}{dt}\left[\frac{1}{2}\ \theta^2\right] = 2\alpha \int_0^\infty k^2 E_\theta\ (k,t)dk \tag{6}$$

3. The Self-Preservation Analysis

We seek self-preserving solutions for which all of the terms in the scalar spectral equations remain in relative balance throughout the decay. We define scaling functions $E_{\theta s}(t)$, $T_{\theta s}(t)$ and $L_\theta(t)$ so that

$$\eta_\theta = kL_\theta(t) \tag{7}$$

$$E_\theta(k,t) = E_{\theta s}(t)f_\theta(\eta_\theta) \tag{8}$$

and

$$T_\theta(k,t) = T_{\theta s}(t)\ g_\theta\ (\eta_\theta) \tag{9}$$

These can be substituted directly into the scalar spectral equation to yield

$$\left[\dot{E}_{\theta s}\right]f_\theta + \left[\frac{E_{\theta s}\dot{L}_\theta}{L_\theta}\right]\eta_\theta f_\theta = \left[T_{s\theta}\right]g_\theta - \left[\frac{\alpha E_{\theta s}}{L_\theta^2}\right]2\eta_\theta^2 f_\theta \tag{10}$$

For self-preservation, all of the bracketed terms must have the <u>same</u> time-dependence. For convenience, the entire expression can be divided by the last bracketed term to yield

$$\left[\frac{\dot{E}_{s\theta}L_\theta^2}{\alpha E_{s\theta}}\right]f_\theta + \left[\frac{\dot{L}_\theta L_\theta}{\alpha}\right]\eta_\theta f'_\theta = \left[\frac{T_{s\theta}L_\theta^2}{\alpha E_{s\theta}}\right]g_\theta - [1]2\eta_\theta^2 f_\theta \tag{11}$$

Now since the last bracketed term is time-independent, all the others must also be time-independent for self-preservation. Therefore, the conditions for self-preservation are

$$\text{(i)} \qquad \frac{\dot{E}_{s\theta} L_\theta^2}{\alpha E_{s\theta}} = \text{constant} \tag{12}$$

$$\text{(ii)} \qquad \frac{L_\theta \dot{L}_\theta}{\alpha} = \text{constant} \tag{13}$$

$$\text{(iii)} \qquad \frac{T_{s\theta} L_\theta^2}{\alpha E_{s\theta}} = \text{constant} \tag{14}$$

Equation (13) can be integrated directly to yield

$$L_\theta^2 \propto \alpha\,(t-t_o) \tag{15a}$$

or

$$L_\theta^2 = 2A\,\alpha\,(t-t_o) \tag{15b}$$

where the origin in time, t=o, is suitably chosen to absorb the initial condition in the <u>mechanical energy equation</u> (see below), A is a coefficient determined by the initial conditions, and the factor of 2 is introduced for convenience. Thus the length scale increases with the square root of time measured from the origin of the scalar field, $t=t_o$.

Equation (12) can be integrated to yield

$$E_{s\theta} = \left[\frac{t-t_o}{t_1-t_o}\right]^q E_{s\theta}\Big|_{t-t_o=t_1-t_o} \tag{16}$$

where t_1 is an arbitrary reference time and q is an exponent determined by the initial conditions. Thus the scalar spectrum undergoes a power law decay.

The spectral decay constant can be related to the decay of the scalar variance by substituting the self-preserving forms into the integral of equation (5) to yield

$$\frac{1}{2}\,\theta^2 = \left[E_{s\theta} L_\theta^{-1}\right] \int_o^\infty f_\theta\,(\eta_\theta)\,d\eta_\theta \tag{17}$$

Thus,

$$E_{s\theta} \sim \theta^2 L_\theta \tag{18}$$

It follows immediately from equations (15)–(18) that

$$\theta^2 \propto (t-t_o)^m \tag{19}$$

where

$$m = q - \frac{1}{2} \tag{20}$$

Thus the scalar variance also undergoes a power law decay with an exponent which is determined by the initial conditions.

The length scale L_θ can be related to a physically meaningful length scale by using the dissipation integral relation. By substituting the self-preserving forms into equation (6) it follows that

$$\epsilon_\theta = \left[\alpha E_{s\theta} L_\theta^{-3}\right] 2 \int_0^\infty \eta_\theta^2 \; f_\theta(\eta_\theta) d\eta_\theta \tag{21}$$

or using equation (18),

$$\epsilon_\theta \propto \alpha\theta^2/L_\theta^2 \tag{22}$$

The scalar Taylor microscale, λ_θ, is defined from (Monin and Yaglom 1975)

$$\overline{\left[\frac{\partial\theta}{\partial x}\right]^2} = \frac{2\theta^2}{\lambda_\theta^2} \tag{23}$$

For an isotropic scalar field the dissipation can be shown to be

$$\epsilon_\theta = 3\alpha \left[\frac{\partial\theta}{\partial x}\right]^2 = 6\alpha \frac{\theta^2}{\lambda_\theta^2} \tag{24}$$

Comparison of equations (23) and (24) makes it immediately clear that

$$L_\theta \propto \lambda_\theta \tag{25}$$

so that the scalar Taylor microscale is proportional to the self-preservation length scale. Therefore without loss of generality we choose the constant of proportionality to be unity (and absorb a factor into f_θ), i.e.

$$L_\theta = \lambda_\theta \tag{26}$$

The constant of proportionality in equation (15b) can be shown to be uniquely related to the scalar variance decay exponent m. From equations (6) and (24),

$$\frac{d\frac{1}{2}\theta^2}{dt} = - \epsilon_\theta = - 6\alpha \frac{\theta^2}{\lambda_\theta^2} \tag{27}$$

By substituting from equations (22) and (18b) for θ^2 and λ_θ it follows that

$$\lambda_\theta^2 = - \frac{12}{m} \alpha (t-t_o) \tag{28}$$

or

$$A = - 6/m. \tag{29}$$

The scalar spectral transfer scaling function $T_{\theta s}$ can be evaluated using equations (14), (18) and (26) as

$$T_{s\theta} \propto \alpha \frac{\theta^2}{\lambda_\theta} \tag{30a}$$

or

$$T_{s\theta} = B_\theta \alpha \frac{\theta^2}{\lambda_\theta} \tag{30b}$$

where B_θ is the constant of proportionality.

Note that there is nothing in the self-preservation analysis which determines the value of the constants. Therefore these will be uniquely determined by the initial conditions. There will be a coupling, however, between the energy decay exponent m, the spectral transfer coefficient B_θ, and the shape of the energy spectrum since these are linked by the spectral energy equation. This coupling can be easily seen by substituting the self-preservation conditions (18), (28), and (30b) into equation (11) with the result that the scalar spectral equation becomes

$$m\left[12f_\theta - 2\eta_\theta^2 f_\theta + B_\theta g_\theta\right] + 6\eta_\theta f_\theta' + 6f_\theta = 0$$

In summary, the scalar spectral equation has been found to admit to self-preserving solutions which have the following properties:

(i) The length scale is the Taylor microscale, λ_θ

(ii) The scalar variance undergoes a power law decay.

(iii) The Taylor microscale grows as the square root of time (measured from its own virtual origin).

(iv) The scalar spectrum and scalar spectral transfer function collapse at all wavenumbers when plotted as

$$\frac{E_\theta(k,t)}{\theta^2\lambda_\theta} \text{ vs. } k\lambda_\theta$$

and

$$\frac{\lambda_\theta T_\theta(k,t)}{\alpha\theta^2} \text{ vs. } k\lambda_\theta$$

(v) The spectral shapes and constants are determined by the initial conditions.

4. The Time Scale and Taylor Microscale Ratios

It follows from equations (6), (19), and the power law decay for the energy that

$$r = m/n \tag{31}$$

where n is the energy decay exponent. Since n and m are determined by the initial conditions, so must be the time scale ratio, and <u>no universal value should be expected.</u>

A fact of immediate interest is that both the mechanical and scalar Taylor microscales grow as the square root of time, although with different virtual origins. Thus asymptotically,

$$\lambda_\theta/\lambda = \text{constant}, \tag{32}$$

the exact value being determined by the initial conditions. The constant in equation (32) can be shown to be given uniquely by the time scale ratio r defined by equation (2). From equations (2), (23) and (3) it follows that

$$r = m/n = 3(\lambda^2/15\nu)/(\lambda_\theta^2/6\alpha)$$

$$= \frac{18}{15}\left[\frac{\alpha}{\nu}\right]\left[\frac{\lambda}{\lambda_\theta}\right]^2 \tag{33}$$

so that

$$\left[\frac{\lambda}{\lambda_\theta}\right]^2 = \frac{5}{6}\left[\frac{m}{n}\right]\sigma = \frac{5}{6}r\sigma \tag{34}$$

where σ is the Prandtl number.

5. The Spectral Transfer and Triple Correlations

As pointed out by George (1987b), a consequence of the spectral transfer scaling (or an equivalent analysis of the von Karman-Howarth equation) is that the triple correlation $\overline{u^2(x)\ u(x+\rho)}$ is given by

$$\overline{u^2(x)\ u(x+\rho)} = R_\lambda^{-1}\ u^3 k(\rho/\lambda) \tag{35}$$

where $k(\rho/\lambda)$ is not the usual non-dimensional triple-correlation introduced by von Karman and Howarth because of the R_λ^{-1} in front of it. Thus the velocity skewness (and the derivative skewness as well) is inversely proportional to R_λ, i.e.

$$\frac{\overline{u^3}}{(\overline{u^2})^{3/2}} = \frac{\overline{u^3}}{u^3} \sim R_\lambda^{-1} \tag{36}$$

A similar relation can be derived from equation (30b) for the velocity-scalar triple correlation (or equivalently, the scalar spectral transfer). Showing,

$$\overline{u(x)\ \theta(x)\ \theta(x+\rho)} = \frac{\alpha}{\lambda_\theta}\ \theta^2\ p(\rho/\lambda_\theta)$$

$$= \left[\frac{\alpha}{\nu}\right]\left[\frac{\nu}{u\lambda}\right]\left[\frac{\lambda}{\lambda_\theta}\right] u\theta^2\ p(\rho/\lambda_\theta) \tag{37}$$

Thus the scalar-velocity triple correlation will depend inversely on the Reynolds number and the scalar to mechanical length scale ratio.

The scalar-velocity triple correlation can be expressed in terms of the time scale ratio r using equation (33) as

$$\overline{u\theta^2}/\sqrt{u^2}\ \theta = (5r/\sigma)^{1/2}\ R_\lambda^{-1} \tag{38}$$

Because $u \sim t^{n/2}$ and $\lambda \sim t^{1/2}$, $R_\lambda \sim t^{(n+1)/2}$; and it follows that

$$\frac{\overline{u\theta^2}}{u\theta^2} \sim t^{-(n+1)/2} \tag{39}$$

Thus the time variation of the scalar velocity triple correlation coefficient depends only on the exponent for the kinetic energy. Since $n < -1$ in all experiments to-date, the normalized triple correlation coefficient increases during decay, just as does that for the velocity alone (George 1987b).

6. Comparison with Experimental Data

The preceeding analysis has predicted with no assumption other than complete self-preservation, that the kinetic energy and mean square thermal variance should decay as some power of the time measured from appropriate virtual origins. Such power law decays have been long established experimentally <u>in the absence of a supporting theory</u>. For example, Figure (1) is reproduced from Wahrhaft and Lumley (1978), and illustrates both the power law decay of the thermal variance and the dependence on the initial conditions of the thermal fluctuations.

A second prediction of the theory is that the mechanical and thermal Taylor microscales should vary as the square root of time (or distance in a wind tunnel) when measured from their respective virtual origins. Since the Taylor microscale, however, is computed from the decay data (eg. using equation 30), and since a power law form for the thermal variance (or kinetic energy) implies directly a square root dependence, the Taylor microscale <u>must</u> behave in the appropriate manner and can therefore not provide independent confirmation. The same is true of relations like equations (32)-(34) relating the time scale ratio to the mechanical and thermal Taylor microscales, since these are satisfied identically once the power law behavior is established for the variances.

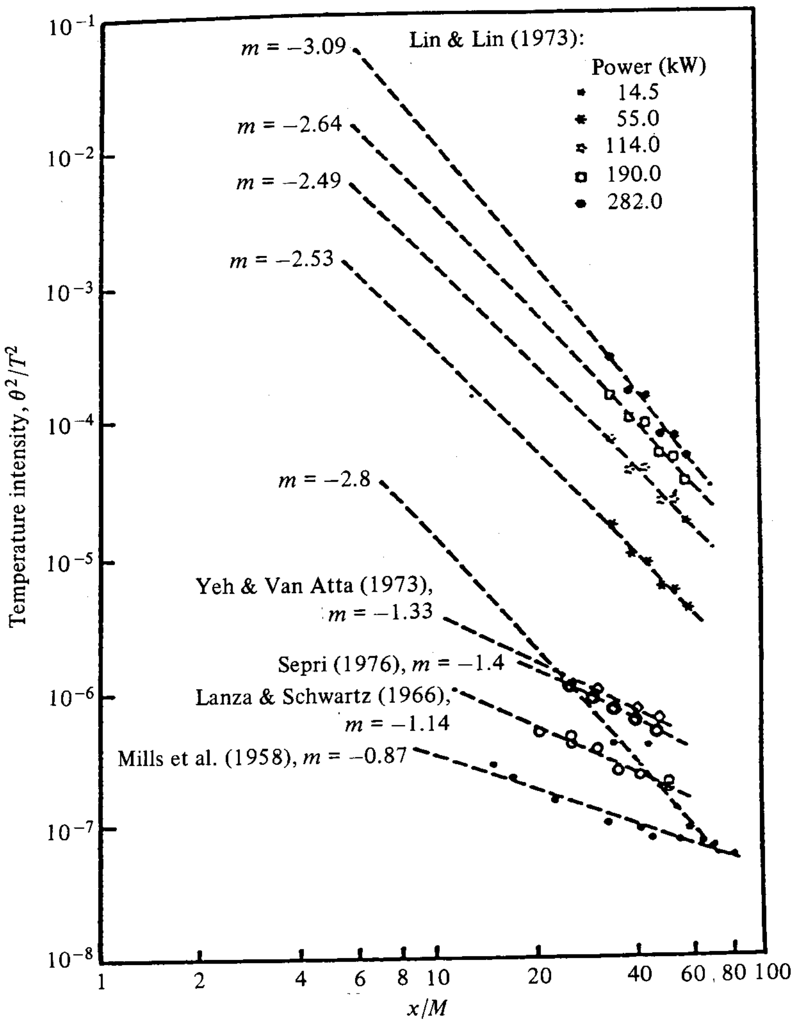

Figure 1. Decay of temperature fluctuations behind heated grid (from Warhaft and Lumley 1978).

The theory predicts that the turbulence can be characterized by a single length scale, the Taylor microscale. Thus, all integral scales must be proportional to it. While this would seem to provide a straightforward test of the theory, unfortunately the integral scale is one of the most difficult parameters to determine experimentally because of the large scales (or low wavenumbers) which determine it. A better experimental test of the proposed scaling laws (than the measured integral scales) is whether the velocity and temperature spectra for a single initial condition can be collapsed throughout the decay. Figures (2) and (3) show the one-dimensional spectral data for the heated grid experiment of Wahrhaft and Lumley (1978, Figures 4 and 10). normalized as $F^1_{11}/u^2\lambda$ versus $k\lambda$ and $F^1_{\theta\theta}$ versus $k\lambda_\theta$ which are the one-dimensional spectral counterparts of the proposed scaling laws. The collapse of the velocity spectral data is over the entire range of scales, including even the largest wavenumbers. This last fact is particularly satisfying in view of the problems cited above since the value of the spectra at the origin can be related directly to the integral scales. The collapse of the temperature spectra is less spectacular but is generally supportive of the theory.

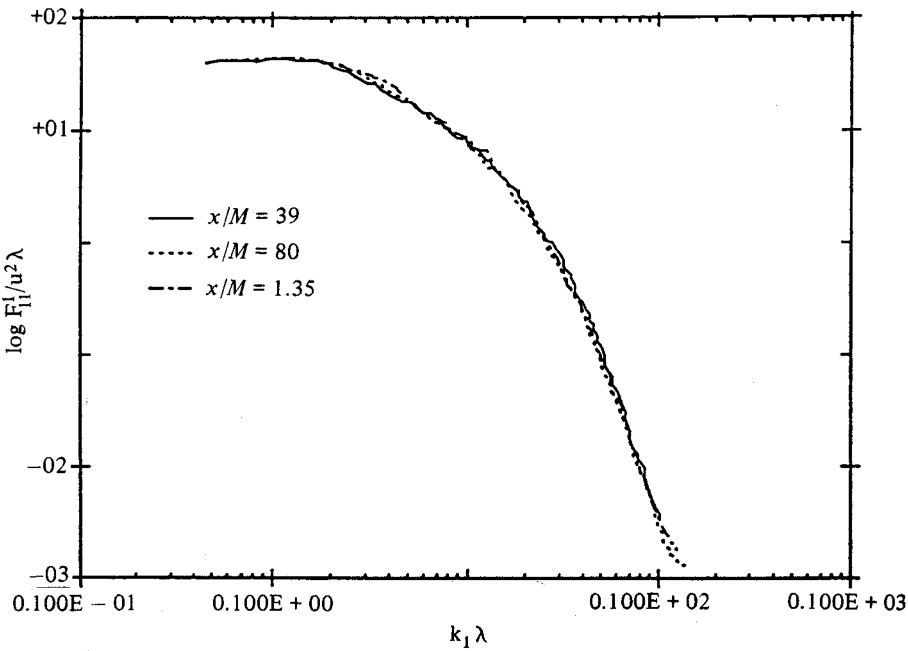

Figure 2. Velocity spectra of Warhaft and Lumley normalized in Taylor variables.

An additional feature of the proposed theory is that each method of generating turbulence will generate its own spectral and decay characteristics which will persist throughout the decay. The entire paper by Wahrhaft and Lumley (1978) is a documentation of this fact and thus this feature must be regarded as confirmed.

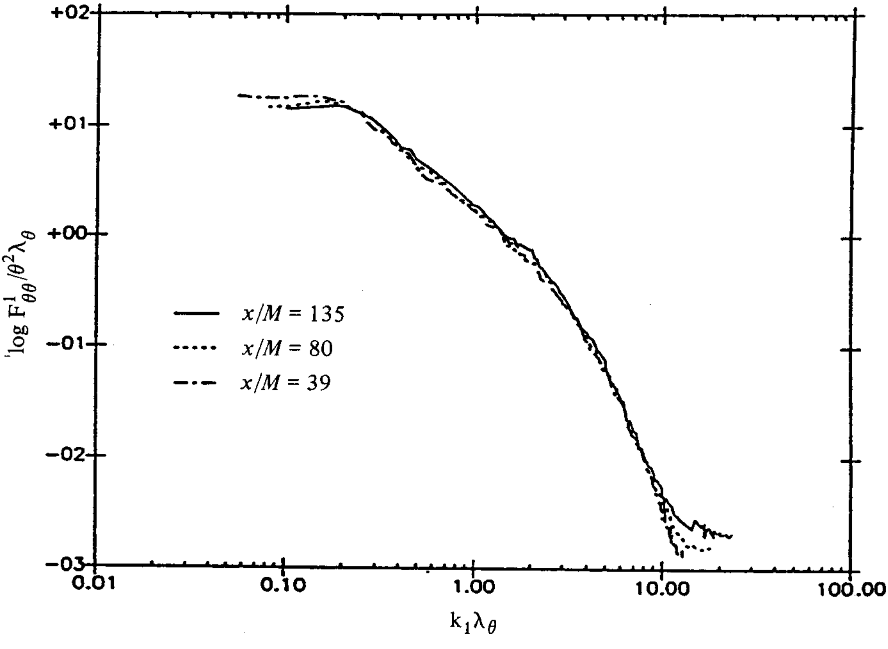

Figure 3. Temperature spectra of Warhaft and Lumley, normalized by Taylor variable.

Finally, the theory predicts that both the mechanical and thermal spectral transfers should scale inversely with the Reynolds number $u\lambda/\nu$. While the downstream development of these does not appear to have been studied, Mills et al. (1958) present measurements of the equivalent triple moments defined by equations (35) and (37), both of which should also show the inverse Reynolds number dependence. These correlations collapse well up to the peak value when plotted as $-p(r)\ R_\lambda$ and $-k(r)R_\lambda$ versus r/λ, but the collapse deteriorates considerably for larger separations, reflecting the same problem as for the double correlations in this experiment. The peak values, however, are in approximate agreement with the proposed scaling as shown in Table I which also includes the velocity-derivative skewness.

TABLE I

Experimental Verification of Predicted Scaling for the Triple Correlations and Velocity Derivative Skewness (from Mills et al. 1958)

x/M	17	32	45	69
$-Re_\lambda k_{max}$	1.48	1.63	–	1.41
$-\sigma Re_\lambda p_{max}$	0.95	1.09	–	1.07
$-Re_\lambda S_u$	10.6	11.0	10.3	9.6

7. Summary and Conclusions

There is enough evidence to indicate that the predicted self-preserving behavior of the decay of temperature fluctuations behind a grid is reasonable. The theory predicts the oft-observed power law dependence for the decay of the temperature variation, the collapse of the spectra, and the observed dependence on the initial conditions. The evidence for the Reynolds number dependence of the non-linear spectral transfer terms is more speculative, and further experimental work would be helpful.

It is interesting that the theory presented here makes no assumptions regarding the relation of the scalar spectral transfer to the mechanical nature of the turbulence. Thus the existing arguments as to the nature of this dependence would appear to be unchanged, as well as the predictions from them (like the k^{-1} range, v. Monin and Yaglom, Vol. II).

On the other hand, it is easy to see that the existence of full self-preservation at all scales of motion negates the possibility of a universal equilibrium range, and with it the validity of the Kolmogorov theory for the university of the small scales (except possibly as the first term in an expansion about the infinite Reynolds number limit, see below). This has been discussed in detail by George (1987a,b) for the mechanical energy spectrum, and the arguments can easily be extended to the scalar spectrum.

A related question has to do with how the self-preservation arguments presented here are applicable in the limit of infinite Reynolds number. The full self-preservation presented here requires that the spectral transfer be forever dependent on Reynolds number. In effect, the spectral transfer adjusts itself so as to provide the required amount of energy dissipation ($\epsilon \sim \nu u^2/\ell^2$) to maintain similarity. This is the opposite of the conventional wisdom which requires that the decay rate be determined only by the energy-containing eddies ($\epsilon \sim u^3/\ell$). The Kolmogorov theory can be recovered in the limit of infinite Reynolds number only if $n \to -1$ in this limit (as speculated by George 1987a).

In summary, there is evidence that the proposed full self-preservation describes many of the observed features of the behavior of isotropic temperature fluctuations. The analysis has raised, however, a number of questions which require additional experimental or numerical simulation research.

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