SOME NEW IDEAS FOR SIMILARITY OF TURBULENT SHEAR FLOWS

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Abstract

The consequences of some new ideas for the similarity of turbulent free shear and boundary layer flows are considered. Unlike earlier the single length and velocity scales of the traditional self-preservation approaches, each statistical quantity is allowed to have its own scale which must be determined from the equations of motion. The equations admit to similarity solutions if such scales can be determined so that all the terms in the equations have the same streamwise dependence. For boundary layer flows, the same ideas are applied to the inner and outer equations separately. Of particular interest is how the solutions depend on the local turbulence Reynolds number.

Keywords: Turbulence, shear flow, similarity, self-preservation, jet, wake, boundary layer, shear layer

1 Introduction

Similarity theories were first applied to turbulence in the 1930's, and have existed as the backbones of turbulence theory since then. Some scientists accept them as gospel, others consider them as simply nice ideas which do not correspond to the world as we find it. This paper attempts to summarize the developments in my own thinking on the subject over the past decade. It was originally intended to be a review, but as so often happens when "reviewing", new ideas evolve from the concentrated effort. Such was the case here, and I have attempted to indicate where that has occurred.

There are three parts to the presentation: First, there is a brief review of the turbulent energy spectrum of isotropic turbulence. It is hoped that the reasons for this will make themselves apparent to the reader. Second, the application of similarity ideas to free shear flows is discussed. Third and finally, an extension of these ideas to turbulent boundary layers is presented. The first of these, the discussion of spectral theory is present in any turbulence book (e.g., Tennekes and Lumley 1972). The applications of the new ideas to free shear flows are primarily drawn from George 1989, while the boundary layer theory is drawn from George et al. 1993 which hopefully will soon be published.

This paper makes no effort, other than in passing, to convince the reader that the ideas are consistent with experimental data. That effort lies in other places, and is largely incomplete. It is my own personal view that there are no data which are not consistent with the new approach. Nonetheless, the state of the data is such that one can use it to justify just about any theory one truly believes in. Therefore the approach here is to focus on the ideas and assumptions which constitute a similarity theory of turbulent shear flows. The emphasis is on building a logically consistent theory of turbulence in which the number of assumptions is minimal and in which they are clearly identified. It is hoped that by doing so, readers will either have to identify the flaws in the proposed theoretical developments, or accept them pending experiments which disagree. At very least it is hoped it will be impossible to ignore them.

A second effort in this paper is to try to identify when the theories of similarity should be expected to work. The classical theories always implicitly depended on the assumption of infinite Reynolds number. This is simply not too useful to experimentalists and numerical modelers who are limited to the finite world. And their efforts to establish the bounds empirically have been less than successful because it was impossible to distinguish between whether the limit was simply not reached, or the theory wrong.

2 The Origin of Reynolds Number Effects

It may seem strange to begin a discussion on similarity of turbulent shear flows with a discussion of the energy spectrum. However, a proper understanding of turbulence energy dissipation is essential to understanding how viscous effects (as measured by the turbulence Reynolds number) enter similarity theory, and dissipation is most easily discussed using the energy spectrum. Now it has been commonly believed that viscosity has nothing to do with similarity theory in turbulent flows since the viscous terms in the averaged equations of motion are negligible. This is true — at least as far as the single point equations for the mean flow are concerned. The Reynolds number of the turbulence, however, has a direct influence on how the dissipation itself scales with respect to the other mean flow quantities: the velocity and length scales in particular. And it is this scale relation which ultimately determines the growth rate of the mean flow, and in some cases whether similarity is possible at all.

The essential elements of viscous dissipation and the energy transfer from scale to scale are present in even isotropic turbulence. Therefore it suffices for the moment to consider only this simplest of turbulent flows. Following Batchelor 1953, the two-point Reynolds stress equations can be Fourier transformed, then integrated over spherical shells of radius k to obtain the spectral energy equation given by

$$\frac{\partial E}{\partial t} = \frac{\partial \varepsilon_k}{\partial k} - 2\nu k^2 E(k) \tag{1}$$

The wavenumber, $k = |\vec{k}|$, can be thought of as the inverse of the "size" of the disturbance. The energy spectrum (or more properly, the three-dimensional spectrum function), E(k), integrated over all wavenumbers yields the total kinetic energy per unit mass, $q^2 = < u^2 + v^2 + w^2 >$; i.e.,

$$\frac{1}{2}q^2 = \int_0^\infty E(k)dk \tag{2}$$

The integral of the last term yields the rate of dissipation of kinetic energy per unit mass (or simply the dissipation), ε; i.e.,

$$\varepsilon = 2\nu \int_0^\infty E(k)dk \tag{3}$$

Figure 1: Energy and dissipation spectra at different Reynolds numbers

The second term is the result of the non-linear interactions of the turbulence and is responsible for the transfer of energy from one scale to another. It has been written so that ε_k is a spectral flux across a given wavenumber. Since this term cannot neither generate nor destroy energy, but only move it around, its integral over all wavenumbers is identically zero.

Figure 1 shows the effect of Reynolds number on the three terms of equation 1. For the lowest Reynolds number, the spectral dissipation is significant in the range where the energy is concentrated. For the highest there is almost no energy dissipation at the wavenumbers containing most of the energy, and little energy at the wavenumbers where the dissipation is occuring. As a consequence, the dissipation is almost entirely dependent on the spectral transfer to move energy from low to high wavenumbers so it can be dissipated. It is easy to see that as the Reynolds number is increased without bound, the dissipation is moved to higher and higher wavenumbers so that in the limit no energy is dissipated directly in the energy-containing range, nor is there any energy in the dissipation range. In fact, if the Reynolds number is high enough, there is an intermediate range called the inertial subrange where there is little energy or dissipation, and energy is simply "passed through" on its way from the low wavenumbers (large scales) to the high wavenumbers (small scales). If so, equation 1 can be split into two equations, one for low wavenumbers given by

$$\frac{\partial E}{\partial t} \approx \frac{\partial \varepsilon_k}{\partial k} \tag{4}$$

and one for high wavenumbers given by

$$0 \approx \frac{\partial \varepsilon_k}{\partial k} - 2\nu k^2 E \tag{5}$$

In the inertial subrange between these limits, the equation is simply

$$0 \approx \frac{\partial \varepsilon_k}{\partial k} \tag{6}$$

In the limit as the turbulence Reynolds number becomes infinite (and only in this limit), the "approximately equal" becomes exactly equal.

Now an immediate consequence of the high wavenumber equations is the famous Kolmogorov 1941 spectral scaling which argues that at very high Reynolds number, the smallest scales of the turbulence are determined mostly by ε and ν. (For example, the only length scale characterizing the dissipative scales is $(\nu^3/\varepsilon)^{1/4}$.) Moreover, the spectral flux, ε_k, must nearly be equal to the dissipation, ε in the inertial subrange. (This is easily seen by integrating both sub-equations over their respective ranges and noting that all of the dissipation is in the highest range.) In fact, in the limit of infinite Reynolds number,

$$\varepsilon_k \equiv \varepsilon \tag{7}$$

in the inertial subrange, and Kolmogorov's approximate scaling becomes exact. Moreover, since viscous effects must be negligible in the inertial subrange, it follows from simple dimensional analysis that $E(k) \sim \varepsilon^{2/3} k^{-5/3}$ there, another famous result due to Kolmogorov.

Now since the integral of the non-linear spectral transfer term must be zero, then it follows from the integral of equation 4 that the low wavenumber range must be entirely determined by only q^2 and ε. Thus, in the limit of infinite Reynolds number (*and only in this limit!*), the length scale characterizing the energy containing scales must be

$$l \equiv \frac{q^3}{\varepsilon} \tag{8}$$

or

$$\varepsilon \equiv \frac{q^3}{l} \tag{9}$$

Note that a length, l, can always be *defined* by equation 8; however, it is an actual physical length in the flow only in the limit of infinite Reynolds number. This is a point of considerable confusion in the turbulence community since the relation $\varepsilon \sim q^3/L$ is often used to estimate the dissipation using a *physical* flow length, L (eg., an integral scale). The key word here is "estimate" since $L \sim l$ only at infinite Reynolds number. This is easily seen by noting that at very low Reynolds numbers where the dissipative and energy ranges overlap, $\varepsilon \sim \nu q^2/L^2$ or $\varepsilon \sim (q^3/L)/(qL/\nu)$. Thus, in general, if L is a *physical length*, then

$$\varepsilon = f\left(\frac{qL}{\nu}\right)\frac{q^3}{L} \tag{10}$$

where f is a function of the turbulence Reynolds number, $R = qL/\nu$, and has limits $f \to 1$ as $R \to \infty$ and $f \to R^{-1}$ as $R \to 0$.

The key to applying similarity theory to real flows lies in understanding the difference between equations 9 and 10. Similarity theories deal ultimately with physical length scales (eg. flow half-widths and boundary layer thicknesses). It will shown below that the growth rate of all simple shear flows ultimately depends how the dissipation scale varies with Reynolds number, the velocity scale, and the length scale. For example, for the axisymmetric jet (and for boundary layers!),

$$d\delta/dx \sim D_s/(U_s^3/\delta) \tag{11}$$

since $q \sim U_s$ and $L \sim \delta$. Thus the growth rate of the jet can be independent of Reynolds number only if $D_s \sim q^3/\delta$. From the above it is clear that this can occur only in the limit of infinite Reynolds number, and in this limit the flow will spread linearly. For finite Reynolds numbers, however, where $\varepsilon \neq q^3/l$, the flow can behave very differently, depending on how the local Reynolds number varies with x. Many of the difficulties in applying the results of similarity theory to actual flows arise from a failure to appreciate the implications of the finite Reynolds number on the dissipation.

3 Free Shear Flows

3.1 Similarity versus Self-Preservation

Turbulent free shear flows have been the favorite application for similarity analysis since the 1930's. The single length and velocity scale hypothesis of von Karman (and made especially popular by Townsend 1956 gave rise to a special "turbulence only" type of similarity analysis called *self-preservation* or *self-preserving flows*. George 1989 pointed out, however, that the single length and velocity scale hypothesis is both unnecessary (to make the problem tractable) and is too restrictive to describe real turbulent flows (except possibly at infinite Reynolds number).

In addition to the single length and velocity scale hypothesis (and often confused with it) there is the idea that shear flows should become asymptotically independent of the details of their initial conditions. In this view, all flows should asymptotically approach their point source equivalents. In fact, as also pointed by George 1989, this idea is not a direct consequence of either a similarity hypothesis or the equations of motion. Rather, it is based on a simple plausibility argument which is not founded in principle. It has been widely accepted by theoreticians because it is believed that it is consistent with experiments. It has been given begrudging acceptance by experimentalists because they believed it to be a theoretical result. In fact, it has presented great problems for experimentalists who have had to perform great manipulative feats to make their data agree with it. The truth is that there is no theoretical justification for it, and little experimental evidence. The vigorous efforts needed by the CFD community to generate results independent of initial conditions is in itself evidence that turbulent flows probably do remember how they started for at least very long times and distances, if not forever.

The approach outlined by George 1989 is essentially the same as that utilized in applying similarity analysis to any laminar flow, with none of the *hocus pocus* of the single length and velocity scale hypothesis. In essence, every dependent variable is allowed to have its own scale. This scale evolves downstream in a manner determined only by the equations of motion, subject to the constraint that all terms in the equations evolve together. This is, of course, quite unlike

Figure 2: The plane (or two-dimensional jet

the single length and velocity scale approach where it is decided quite arbitrarily that the Reynolds stress should scale as the square of the velocity scale (i.e., $R_s = U_s^2$, c.f. Tennekes and Lumley 1972). The example below illustrates the approach for a plane turbulent jet; other examples may be found in George 1989.

3.2 The Plane Jet: The Equations Governing the Mean Flow

Consider the plane jet created by a source of momentum (as well as mass and energy) exhausting into an infinite space. Considerations of the relative order of magnitude of the various terms reduce the averaged momentum equation to

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = -\frac{\partial < uv >}{\partial y} + \{\frac{\partial}{\partial x}(< v^2 > - < u^2 >)\} \tag{12}$$

Note that the appearance of v^2 in the bracketed term results from integration of the y-momentum equation to eliminate the mean pressure. Both the terms in brackets are of second order in turbulence intensity (u'/U) and are generally considered negligible. They can, however, make a slight contribution to the momentum integral equation obtained by integrating equation 12 over all values of y, i.e.,

$$\frac{d}{dx}\int_{-\infty}^{\infty}[U^2 + (< u^2 > - < v^2 >)]dy \tag{13}$$

Integration from the source of the motion to x yields

$$\int_{-\infty}^{\infty}[U^2 + (< u^2 > - < v^2 >)]dy = M_o \tag{14}$$

where M_o is the rate at which kinematic momentum per unit length is added at the source. Thus, regardless of how the plane jet begins, the momentum integral is constant at M_o. (Note that Schneider (1985) has shown that the nature of the entrained flow in the neighborho of the source can modify this value, especially for plane jets. Also the second order ter $< u^2 > - < v^2 >$, while important for accounting for the momentum balance in experimer can be neglected in the analysis below with no loss in generality.

The averaged continuity equation can be integrated to eliminate V in the momentum equ tion, i.e.,

$$V = -\int_0^y \frac{\partial U}{\partial x} d\tilde{y} \quad (\ldots$$

The search for similarity solutions to the averaged equations of motion is exactly like t approach utilized in laminar flows (eg. Batchelor 1967) except that here there are more u knowns than equations, i.e. the averaged equations are not closed. Thus solutions are soug which are of the form

$$U = U_s f(\eta, *)$$

$$-< uv > = R_s g(\eta, *)$$

where

$$\eta = y/\delta$$

The argument "$*$" is used to indicate a possible dependence on source conditions (like e profile, source Reynolds number, etc.). The scale functions are function of x only, i.e $U_s = U_s($ $R_s = R_s(x)$, and $\delta = \delta(x)$ only. Note that the first point of departure from the traditio turbulence analyses has been reached when it was *not* arbitrarily decided that $R_s = U_s^2$, e (c.f. Tennekes and Lumley 1972, Townsend 1956).

Differentiation using the chain-rule, substitution into the averaged momentum equatio and clearing terms yields

$$-\left[\frac{\delta}{U_s}\frac{dU_s}{dx}\right] f^2 - \left[\frac{\delta}{U_s}\frac{dU_s}{dx} + \frac{d\delta}{dx}\right] f' \int_0^\eta f(\tilde{\eta})d\tilde{\eta} = \left[\frac{R_s}{U_s^2}\right] g'$$

Now, all of the x-dependence is in the square bracketed terms, so similarity solutions are possil only if all the bracketed terms have the same x-dependence, i.e.,

$$\frac{d\delta}{dx} \sim \frac{\delta}{U_s}\frac{dU_s}{dx} \sim \frac{R_s}{U_s^2}$$

The coefficients of proportionality can at most depend on the details of how the flow began

Substitution into the momentum integral yields to first order,

$$AU_s^2\delta = M_o$$

where

$$A \equiv \int_0^\infty f^2(\eta, *)d\eta$$

A can at most depend on how the flow began (since every other dependence has been elin nated). Since M_o is independent of x, then

$$U_s \sim M_o^{1/2}\delta^{-1/2}$$

The *local* Reynolds number can be defined as

$$R \equiv U_s \delta / \nu \tag{24}$$

It follows that for the plane jet $R \sim \delta^{1/2}$. Thus, unlike the axisymmetric jet or the plane wake where the local Reynolds number is constant, for the plane jet it will continuously increase as the jet spreads. As noted above and shown below, this has important implications for $d\delta/dx$.

It is easy to show from equation 23 that all of the similarity relations involving U_s are proportional to $d\delta/dx$. Thus the mean momentum equation reduces to

$$-\frac{1}{2}\left\{f^2 - f' \int_0^\eta f(\tilde{\eta})d\tilde{\eta}\right\} = \left[\frac{R_s}{U_s^2}\left(\frac{d\delta}{dx}\right)^{-1}\right] g' \tag{25}$$

The only remaining necessary condition for similarity is

$$\frac{R_s}{U_s^2} \sim \frac{d\delta}{dx} \tag{26}$$

Note that, unlike the similarity solutions encountered in laminar flows, it is possible to have a jet which is similar *without* having some form of power law behavior. In fact, the x-dependence of the flow may not be known at all because of the closure problem. Nonetheless, the profiles will collapse with the local length scale. In fact, it is easy to show that the profiles for all plane jets will be alike if normalized properly, even if the growth rates are quite different! The scale velocity U_s can be defined to be the centerline velocity U_c by absorbing an appropriate factor into the profile function, $f(\eta, *)$. Also, the entire factor in brackets on the right-hand side of equation 25 can be absorbed into the Reynolds stress profile, $g(\eta, *)$. Finally, if the length scale is chosen the same for all flows under consideration (e.g., the half-width, $\delta_{1/2}$, defined as the distance between the center and where $U = U_c/2$), then the similarity equation governing all jets reduces to,

$$f^2 + f' \int_0^\eta f(\tilde{\eta})d\tilde{\eta} = -2g' \tag{27}$$

where $f(0) = 1$ and $g(\eta)$ has the factor $R_s/(U_s^2 d\delta/dx)$ absorbed into it. Thus the mean equation governing all plane jets is the same, regardless of initial conditions! As a consequence, all will have the same similarity velocity profile (when scaled with U_c and $\delta_{1/2}$). This will not be true for the Reynolds stress when plotted as $- < uv > /U_c^2$ (as is usually done) because of the scale factor. Thus the collapse of mean velocity profiles for different source conditions does not indicate a universal state, unless the Reynolds stress also collapses. The difficulties most experimenters have in collapsing Reynolds stress data is because of the failure to consider the scale factor for it.

It is immediately obvious from equation 25 that the usual arbitrary choice of $R_s = U_s^2$ considerably restricts the possible solutions to those for which $d\delta/dx = constant$, or plane jets which grow linearly. (Note that there is nothing in the theory presented here to this point to argue that it does not.) However, even if the growth were linear under some conditions, there is nothing in the theory to this point which suggests the constant should be universal and independent of how the jet began.

The idea that jets might all be described by the same asymptotic growth rate stems from the idea of a jet formed by a line source of momentum only, say ρM_o. Such a source must be infinitesimal in width, since any finite size will also provide a source of mass flow. In the absence of scaling parameters other than distance from the source, x, the only possibilities are $U_s \sim (M_o/x)^{1/2}$, $R_s \sim M_o/x$, and $\delta \sim x$. Obviously, if it has already been assumed that $R_s = U_s^2$, then it makes sense to argue that the constants of proportionality are "universal" since there is only one way to create a line source jet. Thus this whole line of argument of universal states is based on a faulty assumption at the outset; namely that the flow is characterized by a single length and velocity scale.

The problem with the line (or point) source idea is not in the idea itself, but that it has been accepted as the asymptotic state of finite source jets. The usual line of argument (eg., Monin and Yaglom 1971) — if one is presented at all — is as follows: The jet entrains mass so that the amount of mass which has been entrained increases with distance from the source. Eventually the entrained mass overwhelms that added at the source. It is then inferred that the latter can be neglected. George 1989 demonstrated for an axisymmetric jet that this is not the case, and the same arguments apply here. Suppose that in addition to momentum, mass is added at the source at a rate of ρm_o — as in all real jets. Now there is an additional parameter to be considered, and as a consequence, an additional length scale given by $L_m = m_0^2/M_o$. Thus the most that can be inferred from dimensional analysis is that δ/x, $U_s x^{1/2}/M_o^{1/2}$ and $R_s x/M_o$ are functions of x/L_m, with no insight offered into what the functions might be.

3.3 The Plane Jet: The Reynolds Stress Equations

George 1989 further argued that some insight into the growth rate of the jet can be obtained by considering the conditions for similarity solutions of the higher moment equations, in particular the kinetic energy equation. The choice of the kinetic energy equation for further analysis was unfortunate since it implicitly assumed that the three components of kinetic energy all scaled in the same manner. This is, in fact, true only if $d\delta/dx = const$, which is certainly not true *a priori* for turbulent shear flows. Therefore, here the individual component equations of the Reynolds stress will be considered (as should have been done in George 1988).

For the plane jet the equation for $< u^2 >$ can be written to first order (Tennekes and Lumley 1972) as

$$U\frac{\partial < u^2 >}{\partial x} + V\frac{\partial < u^2 >}{\partial y} = 2 < p\frac{\partial u}{\partial x} > + \frac{\partial}{\partial y}\{- < u^2 v >\} - 2 < uv > \frac{\partial U}{\partial y} - 2\epsilon_u \quad (28)$$

where ϵ_u is the energy dissipation rate for $< u^2 >$.

By considering similarity forms for the new moments like

$$\frac{1}{2} < u^2 > = K_u(x)k(\eta) \quad (29)$$

$$< p\frac{\partial u}{\partial x} > = P_u(x)p_u(\eta) \quad (30)$$

$$-\frac{1}{2} < u^2 v > = T_{u^2v}(x)t(\eta) \quad (31)$$

$$\epsilon_u = D_u(x)d(\eta) \quad (32)$$

and using $R_s = U_s^2 d\delta/dx$, it is easy to show that similarity of the $< u^2 >$-equation is possible only if

$$K_u \sim U_s^2 \quad (33)$$

$$P_u \sim \frac{U_s^3}{\delta}\frac{d\delta}{dx} \tag{34}$$

$$T_{u^2v} \sim U_s^3\frac{d\delta}{dx} \tag{35}$$

$$D_u \sim \frac{U_s^3}{\delta}\frac{d\delta}{dx} \tag{36}$$

All of these are somewhat surprising: The first (even though a second moment like the Reynolds stress) because the factor of $d\delta/dx$ is absent; the second, third and fourth because it is present.

Similar equations can be written for the $< v^2 >$, $< w^2 >$, and $< -uv >$-equations; i.e.

$$\begin{aligned} U\frac{\partial < v^2 >}{\partial x} &+ V\frac{\partial < v^2 >}{\partial y} = 2 < p\frac{\partial v}{\partial y} > \\ &+ \frac{\partial}{\partial y}\{- < v^3 > -2 < pv >\} - 2\epsilon_v \end{aligned} \tag{37}$$

$$\begin{aligned} U\frac{\partial < w^2 >}{\partial x} &+ V\frac{\partial < w^2 >}{\partial y} = 2 < p\frac{\partial w}{\partial z} > \\ &+ \frac{\partial}{\partial y}\{- < w^2 v >\} - 2\epsilon_w \end{aligned} \tag{38}$$

$$\begin{aligned} U\frac{\partial < uv >}{\partial x} &+ V\frac{\partial < uv >}{\partial y} =< p\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) > \\ &+ \frac{\partial}{\partial y}\{- < uv^2 >\} - < v^2 > \frac{\partial U}{\partial y} \end{aligned} \tag{39}$$

When each of the terms in these equations is expressed in similarity variables, the resulting similarity conditions are:

$$D_v \sim P_v \sim \frac{U_s K_v}{\delta}\frac{d\delta}{dx} \tag{40}$$

$$D_w \sim P_w \sim \frac{U_s K_w}{\delta}\frac{d\delta}{dx} \tag{41}$$

$$T_{v^3} \sim \frac{U_s K_v}{\delta}\frac{d\delta}{dx} \tag{42}$$

$$T_{w^2v} \sim \frac{U_s K_w}{\delta}\frac{d\delta}{dx} \tag{43}$$

and the real surprise,

$$K_v \sim U_s^2\left(\frac{d\delta}{dx}\right)^2 \tag{44}$$

There is an additional equation which must be accounted for; namely that the sum of the pressure strain-rate terms in the component energy equations be zero (from continuity). Thus, in similarity variables,

$$P_u(x)p_u(\eta) + P_v(x)p_v(\eta) + P_w(x)p_w(\eta) = 0 \tag{45}$$

This can be true for all η only if

$$P_u \sim P_v \sim P_w \tag{46}$$

An immediate consequence is that

$$D_u \sim D_v \sim D_w \tag{47}$$

From equations 34, 40 and 41 it follows that the constraint imposed by 46 can be satisfied only if

$$K_u \sim K_v \sim K_w \tag{48}$$

But from equation 44, this can be true only if

$$\frac{d\delta}{dx} = constant \tag{49}$$

The relations given by equations 48 and 47 were assumed without proof in the George 198 analysis. The additional constraint imposed by equation 49 was not derived, however, and arises from the additional information provided by the pressure strain-rate terms.

Hence, similarity solutions of the Reynolds stress equations are possible only if

$$D_s(x) \sim \frac{U_s^3}{\delta} \tag{50}$$

It is an immediate consequence of the earlier discussion on the nature of the dissipation that there are only two possibilities for this to occur:

i) Either the local Reynolds number of the flow is constant so that the effect of the dissipation on the energy containing eddies (and those producing the Reynolds stress as well) does not vary with downstream distance; or

ii) The local turbulence Reynolds number is high enough so that the relation $\varepsilon \sim q^3/L$ is approximately valid (for a *physical* length $L \sim \delta$!).

Unlike some flows (like the axisymmetric jet or plane wake) where the local Reynolds number is constant, for the plane jet it varies with downstream distance. Therefore the only possibility for similarity at the level of the Reynolds stresses is (ii). This can occur only when the turbulence Reynolds number is large enough, typically 10^4. Since the local Reynolds number for the plane jet continues to increase with increasing downstream distance, this state will eventually be reached. The higher the source Reynolds number, the closer to the exit plane the similarity of the moments will be realized.

3.4 Other Free Shear Flows

The plane jet is but one of the flows which can be analyzed in the manner described above. A few of the possibilities which have already been analyzed are the axisymmetric jet, plane and axisymmetric wakes (George 1989, Hussein et al. 1994). Other possibilities include free shear layers, thermal plumes and the self-propelled wake to mention but a few. All of these fall into the two categories described above: Flows which evolve at constant Reynolds number, and flows which do not. The axisymmetric jet and the plane wake are of the former type, and hence when properly scaled (using the techniques described above) will yield Reynolds number and source dependent solutions. These have been already discussed in detail in the cited papers and will not be discussed further here. The second type of flows (those for which the local Reynolds number varies with streamwise distances) also fall into two types: Those for which the local Reynolds number increases downstream (like the plane jet, plume or the shear layer), and those for which it decreases (like the axisymmetric wake).

When the Reynolds number is increasing with x, the flow will eventually reach the state of full similarity where all of the mean and turbulence quantities collapse. This state will be characterized by the infinite Reynolds number dissipation relation $\varepsilon \sim q^3/\delta$ which will be manifested in the approach of $d\delta/dx$ to its asymptotic value. (This has been shown above to be constant for the plane jet, but will be different for wakes, for example.) Generally this approach will coincide with a turbulent Reynolds number of $q^4/\varepsilon\nu \sim 10^4$ and the emergence in the spectra of the $k^{-5/3}$ range. Before this, the lack of collapse will be most evident in those quantities which depend directly on $d\delta/dx$, like $< -uv >$, $< v^2 >$, etc. Other quantities like the mean flow will collapse much earlier, and as noted above will collapse to profiles independent of source conditions. The latter will not be the case for the second moment quantities since a dependence in the asymptotic value of $d\delta/dx$ due to the source conditions will result in differences in the Reynolds stress equations themselves.

Perhaps the most troubling (and for that reason the most interesting) flows are those where the local Reynolds number is decreasing — like the axisymmetric wake. In these cases, the mean velocity profiles will collapse, assuming the Reynolds number to be large enough that the viscous terms in the mean momentum equation are negligible. The asymptotic growth rate (corresponding to $\varepsilon \sim q^3/\delta$ or in the case of the axisymmetric wake, $\delta \sim x^{1/3}$) will only be achieved as long as the local Reynolds number is high enough (again $q^4/\varepsilon\nu \sim 10^4$). As soon as it drops below this value, the growth rate and scale parameters will begin to deviate (perhaps substantially) from the asymptotic power law forms. The turbulence quantities will begin to reflect this in the lack of collapse — again first noticable in quantities with v^2, etc. which have a direct dependence on $d\delta/dx$. The mean velocity profile, however, will continue to collapse when scaled in local variables. The same will be true for flows in which the source Reynolds number is not high enough for the flow to ever achieve the requisite turbulence Reynolds number — the mean velocity will collapse even though the x-dependence will be all wrong — at least if asymptotic behavior is expected.

4 Boundary Layer Flows

4.1 The Asymptotic Invariance Principle

The problem with wall bounded flows is well-known and will be reviewed briefly below; namely that the presence of the boundary and the no-slip condition imposed there forces the viscous terms back into the mean flow equations, at least near the wall. The traditional approach to the boundary layer equations (eg., Clauser 1954) has been to abandon the possibility of full similarity at the outset, and seek instead local similarity solutions to inner and outer equations. The local similarity solutions obtained are not really similarity solutions of any set of equations, although the scaling arguments from which they are derived presume to capture the physics of the problem in the respective regions of the flow.

An alternative approach (which does not seem to have been previously attempted, at least before George et al. 1993) is to seek full similarity solutions of the inner and outer equations separately. Since these equations are themselves exactly valid only in the limit of infinite Reynolds number, then their full similarity solutions can at most be exactly valid only in this limit. Seen another way, since the equations themselves have neglected terms which are Reynolds number dependent and lose these terms only in the infinite Reynolds number limit,

Figure 3: The turbulent boundary layer

solutions to them will likewise be Reynolds number dependent and lose this dependence only at infinite Reynolds number. This idea will be referred to as the *Asymptotic Invariance Principle.*

The Asymptotic Invariance Principle (although not called by this name) has always been applied to turbulent free shear flows, since similarity solutions for those flows (when they exist) are infinite Reynolds number solutions because the equations from which they are derived are strictly valid only at infinite Reynolds number. The difference in application here is that for the boundary layer there will be two sets of solutions — one which reduces to a full similarity solution of the outer equations at infinite Reynolds number, and another which reduces to a full similarity solution of the inner equations in the same limit. For finite Reynolds numbers, the Reynolds number dependence of the equations themselves, however weak, dictates that the solutions can not be similarity solutions anywhere. But, as noted above, this is no different than for free shear flows which only asymptotically show Reynolds number independence.

In the following sections, the Asymptotic Invariance Principle will be applied to some of the single point equations governing the zero pressure gradient turbulent boundary layer. In particular, solutions will be sought which reduce to full similarity solutions of the equations in the limit of infinite Reynolds number, first for the inner equations and then for the outer. The form of these solutions will determine the appropriate scaling laws for finite as well as infinite Reynolds number, since alternative scaling laws could not be independent of Reynolds number in the limit. Once the method has been established by application to the equations governing the mean momentum, then the same principle will be applied to equations governing the Reynolds stresses and the statistical quantities appearing in them, just as for the free shear flows described earlier.

4.2 Governing Equations and Boundary Conditions

The equation of motion and boundary conditions appropriate to a zero-pressure gradient turbulent boundary layer (with constant properties) at high Reynolds number are well-known to be given by (Tennekes and Lumley 1972)

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = \frac{\partial}{\partial y}\left[- < uv > + \nu\frac{\partial U}{\partial y}\right] \tag{51}$$

where $U \rightarrow U_\infty$ as $y \rightarrow \infty$ and $U = 0$ at $y = 0$.

The presence of the no-slip condition precludes the possibility of similarity solutions (at least for the entire boundary layer), and so solutions are sought which asymptotically (at infinite Reynolds number) satisfy the following outer and inner equations and boundary conditions:

- Outer Region

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = \frac{\partial}{\partial y}[- < uv >] \tag{52}$$

  where $U \rightarrow U_\infty$ as $y \rightarrow \infty$.

- Inner (or near wall) region

$$0 = \frac{\partial}{\partial y}\left[- < uv > + \nu\frac{\partial U}{\partial y}\right] \tag{53}$$

  where $U \rightarrow 0$ at $y = 0$.

The neglected terms in both inner and outer equations vanish identically only at infinite Reynolds number. However, there is nothing in the development of these equations which precludes their approximate validity from the time the flow begins to develop unsteady disturbances, as long as the Reynolds number is large.

Equation 53 for the inner region can be integrated directly to obtain

$$- < uv > + \nu\frac{\partial U}{\partial y} = \frac{\tau_W}{\rho} \equiv u_*^2 \tag{54}$$

where τ_W is the wall shear stress and u_* is the corresponding friction velocity defined from it. It is clear that in the limit of infinite Reynolds number (but only in this limit) that the total stress is constant across the inner layer, and hence its name "the Constant Stress Layer". It should be noted that the appearance of u_* in equation 54 does not imply that the wall shear stress is an independent parameter (like ν or U_∞). It enters the problem only because it measures the forcing of the inner flow by the outer; or alternatively, it can be viewed as measuring the retarding effect of the inner flow on the outer. Thus u_* is a *dependent* parameter which must be determined by matching solutions of the inner and outer equations.

It is also interesting to note that the inner layer occurs only because of the necessity of including viscosity in the problem so that the no-slip condition can be met. The outer layer, on the other hand, is dominated by inertia and the effects of viscosity enter only through the matching to the inner layer. Thus the outer flow is effectively governed by inviscid equations, *but with viscous-dominated inner boundary conditions set by the inner layer.*

4.3 The Velocity Scaling Laws

Before applying the new similarity approach (the AIP) to the zero-pressure gradient boundary layer, it is useful to review how the more traditional approach of *local similarity* developed and what conclusions were drawn from it. The phrase local similarity or local self-preservation which is commonly used to describe these traditional approaches (c.f. Clauser 1954, Tennekes and Lumley 1972) really means that the investigator suspects that there should be similarity solutions even though the equations do not appear to admit to them. As shown below, these *local* solutions are arrived at primarily by dimensional and physical inspection, and not by rigorous analysis. The AIP proposed above has some of the same features, but adds the additional constraint that the *local* solutions must reduce to full similarity solutions of the inner and outer equations separately in the limit of infinite Reynolds number. Solutions to the governing equations are sought which depend only on the streamwise coordinate through a local length scale $\delta(x)$. Thus, for the mean velocity

$$U = U(y, \delta, U_\infty, u_*, \nu) \tag{55}$$

It should be noted that u_* cannot be viewed as an independent parameter since it is determined once U_∞, ν and x (or δ) are given. This dependence can be expressed from dimensional considerations as a friction law by

$$\frac{u_*}{U_\infty} = \sqrt{\frac{c_f}{2}} = g(\epsilon) \tag{56}$$

where

$$\epsilon \equiv \frac{\nu}{u_* \delta} \tag{57}$$

$$c_f \equiv \frac{\tau_w}{\frac{1}{2}\rho U_\infty^2} = 2\frac{u_*^2}{U_\infty^2} \tag{58}$$

Application of the Buckingham Pi theorem to the velocity itself yields a number of possibilities, *all of which describe the variation of the velocity across the entire boundary layer.* Among them are:

$$\frac{U}{u_*} = f_i\left[\frac{yu_*}{\nu}, \epsilon\right] \tag{59}$$

$$\frac{U - U_\infty}{U_\infty} = f_o\left[\frac{y}{\delta}, \epsilon\right] \tag{60}$$

$$\frac{U - U_\infty}{u_*} = F_o\left[\frac{y}{\delta}, \epsilon\right] \tag{61}$$

Note that since u_*/U_∞ and ϵ (or $u_*\delta/\nu$) are related by equation 56, either can be retained (and the other omitted) in equations 59 — 61 with no loss of information.

In the limit as $\epsilon \to 0$ (or equivalently, $u_*\delta/\nu \to \infty$ or $u_*/U_\infty \to 0$), equation 59 becomes asymptotically independent of δ and U_∞, and thus can at most describe a limited region very close to the wall, i.e.,

$$\frac{U}{u_*} = f_{i\infty}\left[\frac{yu_*}{\nu}\right] \tag{62}$$

This is, of course, the familiar *Law of the Wall* expressed in inner variables as originally proposed by Prandtl (1932)

A similar limiting argument for f_o and F_o yields two quite different candidates for an outer profile; namely,

$$\frac{U - U_\infty}{U_\infty} = f_{o\infty}\left(\frac{y}{\delta}\right) \tag{63}$$

and

$$\frac{U - U_\infty}{u_*} = F_{o\infty}\left(\frac{y}{\delta}\right) \tag{64}$$

Both cannot, of course, be Reynolds number independent (and finite) in the limit since the ratio u_*/U_∞ continues to vary.

The first form given by equation 63 has only been fleetingly considered by the fluid dynamics community, and discarded in favor of the second alternative. Millikan, for example, appears to have considered it briefly, noted that it leads to self-preserving power law solutions of the outer equations, and then dismissed these solutions as interpolation formulas. Clauser 1954 (see also Hinze 1975) plotted only the highest and lowest Reynolds number data of Schultz-Grunow 1941 in deficit form, and concluded that the collapse using equation 63 was not as satisfactory as that obtained using the deficit form of equation 64. There is no evidence that either of these conclusions has been refuted, or even questioned.

The second form given by equation 64 is the traditional choice. It was originally used by Stanton and Pannell 1914 for pipe flows and adapted by von Karman 1930 to the boundary layer. Millikan 1938 matched the 'inner' scaling of equation 62 to the 'outer' scaling of equation 64 in the limit of infinite Reynolds number to obtain the familiar inertial sublayer profiles as

$$\frac{U}{u_*} = \frac{1}{\kappa} ln\left(\frac{y}{\eta}\right) + B \tag{65}$$

$$\frac{U - U_\infty}{u_*} = \frac{1}{\kappa} ln\left(\frac{y}{\delta}\right) + B_1 \tag{66}$$

and a friction law given by

$$\frac{U_\infty}{u_*} = \sqrt{\frac{2}{c_f}} = -\frac{1}{\kappa} ln\,(\varepsilon) + (B - B_1) \tag{67}$$

where κ, B, and B_1 are presumed to be universal constants. (Actually Millikan only analyzed the pipe and channel flows, but indicated that a boundary layer application would follow. One can presume that this second paper was the 1954 Clauser paper.)

By substituting the inner and outer scaling laws into the defining integrals for the displacement and momentum thicknesses, Clauser 1954 showed that

$$\frac{\delta_*}{\delta} = A_1 \frac{u_*}{U_\infty} \tag{68}$$

$$\frac{\theta}{\delta} = A_1 \frac{u_*}{U_\infty}\left[1 - A_2 \frac{u_*}{U_\infty}\right] \tag{69}$$

where A_1 and A_2 are universal constants which can be evaluated from integrals of the outer velocity profile function. It follows that the shape factor is given by the asymptotic relation

$$H = \frac{\delta_*}{\theta} = 1 - A_2 \frac{u_*}{U_\infty} \tag{70}$$

Thus as $\varepsilon \to 0$ and $u_*/U_\infty \to 0$, $H \to 1$.

The underlying assumption of the above matching is that the inner and outer scaling laws used for the profiles, in fact, have a region of common validity (or overlap) in the limit as $\epsilon \to 0$ or $u_*/U_\infty \to 0$. Long and Chen 1981 have remarked that it is strange that the matched layer between one characterized by inertia and another characterized by viscosity does not depend on both inertia and viscosity, but only inertia (hence the term 'inertial sublayer', Tennekes and Lumley 1972). They further suggested that this might be a consequence of improperly matching two layers which did not overlap. The fact that the limiting ratio of the outer length scale δ to both of the commonly used integral length scales, δ_* and θ, is infinite lends considerable weight to their concern. In particular, this implies that from the perspective of the outer flow, the boundary layer does not exist at all in the limit of infinite Reynolds number. If one imagines approaching this limit along a semi-infinite plate where the boundary layer continues to grow, the outer length scale increases faster than any dynamically significant integral length. This is particularly troubling since δ itself is unspecified by the theory and can not be related to physically measurable length scales except through the degenerate expressions above.

Theoretical objections notwithstanding, there has been widespread acceptance (canonization might be more accurate) of the Millikan/ Clauser theory because it is believed to be consistent with the experimental data. This has never been entirely true, and as better data have been acquired it has become even more evident to be false. Even Coles 1962, whose careful determinations of the constants are the most often cited, expressed puzzlement at the apparent failure of the outer velocity profile (the "wake" constant in particular) to achieve Reynolds number independence when scaled with u_*. The recent careful review of Gad-el-Hak and Bandyopadyhay 1994 lists a number of experiments where persistent Reynolds number trends in the mean profile deficit are observed, even at relatively high Reynolds number. And there have been persisent and nagging problems in trying to reconcil direct measurements of wall shear stress (either by sensors on the wall or direct measurements in the linear layer) with the shear stress inferred from the logarithmic region using the "accepted" constants, the so-called Clauser method. Finally, when the same scaling arguments are extended to the higher order turbulence moments (second moments and above), they fail to collapse the data outside of the viscous sublayer ($y^+ > 10$ or so) (v. Gad-el-Hak and Bandyopadyhay 1994).

In view of the above, it is useful to examine whether and why u_* should be a scaling parameter for the outer flow at all, especially since there is an alternative which does not use it. First note that it is only in the limit of infinite Reynolds number where the inner layer is truly a constant stress layer. Thus, only in this limit is the shear stress experienced by the outer flow exactly measured by u_*^2. At all finite Reynolds numbers it only approximately measures the effect of the inner layer on the outer. While the use of u_* as an outer scaling parameter may give reasonable results over a rather large range of Reynolds numbers, it can not be an appropriate choice for the cornerstone of an asymptotic analysis of the outer boundary layer. (This has also been pointed out by Panton 1990 who tries to "fix" the problem with a higher order analysis while retaining the same deficit law.) This is in contrast with fully-developed turbulent pipe or channel flow where the overall balance between pressure and viscous forces on a section of the flow dictates that both the inner and outer flow scale with u_*, a direct consequence of the streamwise homogeneity. An obvious consequence of these observations is that the wall layers of these homogeneous flows are fundamentally different from those of the inhomogeneous boundary layer, contrary to popular belief (c.f. Monin and Yaglom 1972. Tennekes and Lumley 1972).

There have been numerous attempts to place the Millikan/Clauser theory on a more secure footing and extend it to higher order, especially notable amoung them: Bush and Fendell 1974, Long and Chen 1981, and Panton 1990. All began with the same velocity deficit, and therefore will not be considered further here. In the remainder of this paper,the alternative formulation of the outer profile given by equation 63 and the Law of the Wall will be shown to follow directly from the hypothesis that the outer and inner flow equations should admit to similarity solutions of the more general form described earlier. The consequences of matching two regions governed by different parameters will be explored, and the governing relations for a variety of turbulence quantities will be derived.

4.4 Full Similarity of the Inner Equations

In keeping with the Asymptotic Invariance Principle set forth above, similarity solutions to the inner equations and boundary conditions are sought which are of the form

$$U = U_{si}(x) f_i(y^+) \quad (71)$$

$$-\overline{uv} = R_{si}(x) r_i(y^+) \quad (72)$$

where

$$y^+ \equiv \frac{y}{\eta} \quad (73)$$

and the length scale η remains to be determined.

Substitution into equation 54 and clearing terms yields to leading order

$$\left[\frac{u_*^2}{U_{si}^2}\right] = \left[\frac{R_{si}}{U_{si}^2}\right] r_i + \left[\frac{\nu}{\eta U_{si}}\right] f_i{}' \quad (74)$$

The choice for η is obviously

$$\eta = \nu / U_{si} \quad (75)$$

from which it follows immediately that similarity solutions are possible only if the inner Reynolds stress scale is given by

$$R_{si} = U_{si}^2 \quad (76)$$

It is now also obvious that the inner velocity scale must be the friction velocity so that

$$U_{si} \equiv u_* \quad (77)$$

It follows that

$$\eta = \nu / u_* \quad (78)$$

$$R_{si} = u_*^2 \quad (79)$$

The integrated inner equation can now be written (to leading order in ε) as

$$1 = r_i + f_i{}' \quad (80)$$

The similarity variables derived above are the usual choices for the inner layer, and thus the law of the wall is consistent with full similarity of the inner equations, *in the limit of infinite Reynolds number*. For any finite (but large Reynolds number) the solutions for inner layer will retain a Reynolds number dependence (as discovered from the Pi-theorem in deriving equation 59) since the governing equations themselves do so. It is obvious that it is equation 59 which reduces to the proper limiting form to be a similarity solution for the inner layer, and thus it is the real Law of the Wall. For finite Reynolds numbers, however, it describes the velocity profile over the entire layer. These ideas are not incompatible, since from the perspective of the inner layer the outer layer is never reached in this limit (i.e., $\delta^+ \to \infty$).

4.5 Full Similarity of the Outer Equations

In accordance with the Asymptotic Invariance Principle, solutions of the outer momentum equation and boundary conditions will be sought which reduce to similarity solutions of those equations in the limit of infinite Reynolds number. It is important to again note that no scaling laws are assumed at the outset, but rather will be derived from the conditions for similarity of the equations.

For the outer equations, solutions are sought which are of the form

$$U - U_\infty = U_{so} f_o(\overline{y}) \tag{81}$$

$$-\overline{uv} = R_{so} r_o(\overline{y}) \tag{82}$$

where

$$\overline{y} = y/\delta \tag{83}$$

and U_{so}, R_{so}, and δ are functions only of x. The velocity has been written as a deficit to avoid the necessity of accounting for its variation across the inner layer. This is, of course, not possible with the Reynolds stress since it vanishes outside the boundary layer. The V-component of velocity has been eliminated by integrating the continuity equation from the wall, thus introducing a contribution from the inner layer which vanishes identically at infinite Reynolds number.

Substitution into equation 52 and clearing terms yields

$$\left[\left(\frac{U_\infty}{U_{so}}\right)\frac{\delta}{U_{so}}\frac{dU_{so}}{dx}\right] f_o + \left[\frac{\delta}{U_{so}}\frac{dU_{so}}{dx}\right] f_o^2 - \left[\frac{U_\infty}{U_{so}}\frac{d\delta}{dx}\right]\overline{y} f_o''$$
$$- \left\{\frac{d\delta}{dx} + \left[\frac{\delta}{U_{so}}\frac{dU_{so}}{dx}\right]\right\} f_o' \int_0^{\overline{y}} f_o(\psi)d\psi = \left[\frac{R_{so}}{U_{so}^2}\right] r_o' \tag{84}$$

It is clear that full similarity (or self-preservation) is possible only if

$$U_{so} \sim U_\infty \tag{85}$$

and

$$R_{so} \sim U_\infty^2 \frac{d\delta}{dx} \tag{86}$$

Thus, if the outer equations admit to similarity solutions, the velocity scale for the velocity deficit law must be U_∞, and not u_* as suggested by Von Karman 1930 and widely utilized since (eg. Clauser 1954, Coles 1956).

The analysis above makes it clear that of the possible candidates for an outer scaling law for the velocity, only the profile represented by equation 60 is Reynolds number invariant in the limit. Therefore this must be the appropriate scaling law for finite Reynolds numbers as well. (This is, of the course, the whole idea behind the Asymptotic Invariance Principle.) On the other hand, the usual deficit profile, equation 61, can not be Reynolds number invariant in the limit. In fact, since the velocity deficit scaled with U_∞ (f_o from equation 60) is Reynolds number invariant in the limit, it is clear why the usual velocity deficit profile scaled with u_* (F_o from equation 61) also vanishes in this limit, since $F_o = (u_*/U_\infty) f_o$ and $u_*/U_\infty \rightarrow 0$.

The Reynolds stress scale, on the other hand, is *not* U_∞^2, but an entirely different scale depending on the growth rate of the boundary layer, $d\delta/dx$. It will be shown below that R_{so} can be determined by matching the outer Reynolds stress to the inner Reynolds stress. The need for such a matching is intuitively obvious, since the only non-zero boundary condition on the Reynolds stress in the outer flow is that imposed by the inner.

Some have objected to the type of similarity analysis employed here as leading to unphysical results for the boundary layer. Certainly there is nothing unphysical about the velocity deficit law using U_∞ in and of itself, and a case for such a deficit law could have been made, even with the data available at the time (as will be shown later). Thus the fundamental basis for this objection must have been the condition on the Reynolds stress. However, this would have been a problem only if it were also required or *assumed at the outset* that $R_{so} = U_{so}^2$, for then it would have also been necessary that $d\delta/dx = constant$. Since the boundary layer was well-known not to grow linearly, Millikan (and many before and after him as well) was forced to conclude that full self-preservation (in the assumed sense) was not possible, and therefore had to settle for a *locally* self-preserving solution.

As pointed out earlier for free shear flows, there is no reason to insist that $R_{so} = U_{so}^2$. If this arbitrary requirement is relaxed, then there is no longer the requirement for linear growth, and both equation 86 and similarity become tenable. In fact, these conditions require that the outer flow be governed by two velocity scales, U_∞ and a second governing the Reynolds stress which is determined by the boundary conditions imposed on the Reynolds stress by the inner layer. It will be shown below that the inner and outer Reynolds stresses can match (to first order) only if

$$U_\infty^2 \frac{d\delta}{dx} \sim u_*^2 \tag{87}$$

which resembles closely the momentum integral equation, both a surprising and gratifying result. More will be said on this relationship later.

That the outer (and inner) equations admit to fully self-preserving solutions (in the sense of George 1989 will come as no surprise to the experimentalists who have long recognized their ability to collapse the outer mean velocity data with only U_∞ and δ, δ_*, or θ. Hinze 1976 and Schlichting 1968, for example, show profiles normalized by U/U_∞, and document the variation with Reynolds number of a power law fit to the entire profile. Even the fact that the outer Reynolds stress scales with u_* (but only to first order) is in accord with common practice, since it is assumed in the old theory — but in a way which could not account for the observed weak dependence on Reynolds number. Thus one can speculate that Millikan's conclusions might have been quite different had he (and several generations after him) not been locked-in to a too restrictive definition of self-preservation.

4.6 The Matched Layer: A New View

It is obvious that since both the outer and inner profiles are non-dimensional profiles with different scales, since the ratio of the scales is Reynolds number dependent any region between the two similarity regimes cannot be Reynolds number independent. However, as was shown earlier from dimensional considerations, both inner and outer forms describe the entire flow if either of the arguments u_*/U_∞ or $\varepsilon = \eta/\delta$ are retained. Therefore at finite Reynolds numbers, both equations 59 and 60 describe the region between the two similarity regimes (and in fact the entire flow field). This is quite different from the usual asymptotic matching where inner and outer solutions are matched in an overlap region. Here both solutions are valid everywhere, at least for finite Reynolds numbers. Hence the objective is *not* to see if they overlap and match them if they do; rather, it is to determine if the fact that they degenerate at infinite Reynolds number in different ways determines their functional forms in the common region they describe.

There are several pieces of information about the two profiles which can be utilized in this determination without further assumptions. They are:

- First, since both inner and outer forms of the velocity profile must describe the flow everywhere as long as $\epsilon = \eta/\delta = 1/\delta^+$ is finite, it follows from equations 59 and 60 that

$$1 + f_o(\overline{y}, \epsilon) = g(\epsilon) f_i(y^+, \epsilon) \tag{88}$$

  Recall that $g(\epsilon)$ is defined by equation 56 and that $g \to 0$ as $\epsilon \to 0$.

- Second, for finite values of ϵ, the velocity derivatives from both forms of the velocity must also be the same everywhere. It is easy to show that this requires that

$$\frac{\overline{y}}{1 + f_o} \frac{df_o}{d\overline{y}} = \frac{y^+}{f_i} \frac{df_i}{dy^+} \tag{89}$$

  for all values of ϵ and y.

- Third, in the limit, however, both f_o and f_i must become asymptotically independent of ϵ. Thus $f_o(\overline{y}, \epsilon) \to f_o(\overline{y})$, and $f_i(y^+, \epsilon) \to f_i(y^+)$ as $\epsilon \to 0$ (or otherwise the velocity scales have been incorrectly chosen). This is, in fact, the *Asymptotic Invariance Principle.*

Now the problem is that in the limit as $\epsilon \to 0$, the outer form fails to account for the behaviour close to the wall while the inner fails to describe the behavior away from it. The question is: In this limit (as well as for all finite values approaching it), does there exist an "overlap" region where equation 88 is still valid? Since both δ and η are increasing with streamwise distance along the surface, this "overlap" region will not only increase in extent when measured in either inner or outer coordinates, it will move farther from the wall in actual physical variables. (Note that this is quite different from pipe and channel flows in which the overlap layer remains at fixed distance from the wall because of the streamwise homogeneity, as long as the external parameters are fixed.)

Because of the movement of the matched layer away from the wall with increasing x, it is convenient and necessary to introduce an intermediate variable $\tilde{y}$ which can be fixed in the overlap region all the way to the limit, regardless of what is happening in physical space (v. Cole and Kevorkian 1981). A definition of $\tilde{y}$ which accomplishes this is given by

$$\tilde{y} = y^+ \epsilon^n \tag{90}$$

or

$$y^+ = \tilde{y} \epsilon^{-n} \tag{91}$$

Since $\overline{y} = y^+ \epsilon$, it follows that

$$\overline{y} = \tilde{y} \epsilon^{1-n} \tag{92}$$

For all values of n satisfying $0 < n < 1$, $\tilde{y}$ can remain fixed in the limit as $\epsilon \to 0$ while $\overline{y} \to 0$ and $y^+ \to \infty$. Substituting these into equation 88 yields the matching condition on the velocity as

$$1 + f_o(\tilde{y} \epsilon^{1-n}, \epsilon) = g(\epsilon) f_i(\tilde{y} \epsilon^{-n}, \epsilon) \tag{93}$$

Now equation 93 can be differentiated with respect to ϵ to yield

$$\left. \frac{\partial(1 + f_o)}{\partial \overline{y}} \right|_\epsilon \frac{\partial \overline{y}}{\partial \epsilon} + \left. \frac{\partial(1 + f_o)}{\partial \epsilon} \right|_{\overline{y}} = \frac{dg}{d\epsilon} f_i + g \left\{ \left. \frac{\partial f_i}{\partial y^+} \right|_\epsilon \frac{\partial y^+}{\partial \epsilon} + \left. \frac{\partial f_i}{\partial \epsilon} \right|_{y^+} \right\} \tag{94}$$

Carrying out the indicated differentiation of y^+ and $\overline{y}$ by ϵ (for fixed $\tilde{y}$), and multiplying by $\epsilon/(1+f_o)$ yields (after some rearranging)

$$(1-n)\frac{\overline{y}}{(1+f_o)}\frac{\partial(1+f_o)}{\partial\overline{y}}\bigg|_\epsilon + n\frac{y^+}{f_i}\frac{\partial f_i}{\partial y^+}\bigg|_\epsilon = \frac{\epsilon}{g}\frac{dg}{d\epsilon} + \left\{\frac{\epsilon}{f_i}\frac{\partial f_i}{\partial\epsilon}\bigg|_{y+} - \frac{\epsilon}{1+f_o}\frac{\partial(1+f_o)}{\partial\epsilon}\bigg|_{\overline{y}}\right\} \tag{95}$$

It follows immediately from equation 89 that

$$\frac{\overline{y}}{1+f_o}\frac{\partial(1+f_o)}{\partial\overline{y}}\bigg|_\epsilon = \frac{\epsilon}{g}\frac{dg}{d\epsilon} + \left\{\frac{\epsilon}{f_i}\frac{\partial f_i}{\partial\epsilon}\bigg|_{y+} - \frac{\epsilon}{1+f_o}\frac{\partial(1+f_o)}{\partial\epsilon}\bigg|_{\overline{y}}\right\} \tag{96}$$

Equation 96 can be usefully rewritten as

$$\frac{\overline{y}}{1+f_o}\frac{\partial(1+f_o)}{\partial\overline{y}}\bigg|_\epsilon = \gamma(\epsilon) + \mu(\epsilon;\overline{y},y^+) \tag{97}$$

where $\gamma = \gamma(\epsilon)$ is defined by

$$\frac{\epsilon}{g}\frac{dg}{d\epsilon} \equiv \gamma(\epsilon) \tag{98}$$

and $\mu = \mu(\epsilon;\overline{y},y^+)$ is defined by

$$\mu \equiv \left\{\frac{\epsilon}{f_i}\frac{\partial f_i}{\partial\epsilon}\bigg|_{y+} - \frac{\epsilon}{1+f_o}\frac{\partial(1+f_o)}{\partial\epsilon}\bigg|_{\overline{y}}\right\} \tag{99}$$

The first term on the right hand side 97 is at most a function of ϵ alone, while the second term contains all of the residual y-dependence.

It is easy to show that $\mu \to 0$ as $\epsilon \to 0$ since f_i and f_o are asymptotically independent of ϵ and finite (since they were determined to be from the Asymptotic Invariance Principle). Thus the first term on the right hand side, γ, dominates in the limit (since $g \to 0$ as $\epsilon \to 0$ and g occurs in the denominator), and the additional terms, μ, represent the contribution of higher order terms (which vanish identically in the limit). Thus, to leading order in ϵ equation 96 can be written as

$$\frac{\overline{y}}{1+f_o}\frac{\partial(1+f_o)}{\partial\overline{y}}\bigg|_\epsilon = \gamma(\epsilon) \tag{100}$$

From equation 89, it also follows that

$$\frac{y^+}{f_i}\frac{\partial f_i}{\partial y^+}\bigg|_\epsilon = \gamma(\epsilon) \tag{101}$$

These can be readily integrated to yield (to leading order in ϵ),

$$1+f_o(\overline{y},\epsilon) = C_o(\epsilon)\overline{y}^{\gamma(\epsilon)} \tag{102}$$

$$f_i(y^+,\epsilon) = C_i(\epsilon)y^{+\gamma(\epsilon)} \tag{103}$$

It follows immediately from equation 88 that

$$g(\epsilon) = \frac{C_o(\epsilon)}{C_i(\epsilon)}\epsilon^{\gamma(\epsilon)} \tag{104}$$

However, equation 98 must also be satisfied. Substituting equation 104 into equation 98 implies that γ, C_o, and C_i are constrained by

$$ln\epsilon\frac{d\gamma}{d\epsilon} = \frac{d}{d\epsilon}ln\left[\frac{Ci}{C_o}\right] \tag{105}$$

The functions $C_i(\epsilon)$, $C_o(\epsilon)$, and γ must still be determined, either from experimental data or a closure model for the turbulence. The presence of the $ln(\epsilon)$ term on the left-hand side makes it clear that whatever the variation of C_i and C_o with ϵ, the variation of γ will be less.

Equation 105 is exactly the criterion for the neglected terms in equation 96 to vanish identically. Therefore the solution represented by equations 102 – 105 is, indeed, the first order solution for the velocity profile in the matched layer at finite, but large, Reynolds number. Second order solutions, say $f_o^{(2)}(\overline{y}, \epsilon)$ and $f_i^{(2)}(y^+, \epsilon)$ must satisfy

$$\frac{\overline{y}}{1+f_o^{(2)}}\frac{\partial(1+f_o^{(2)})}{\partial\overline{y}}\Bigg|_\epsilon = \mu \tag{106}$$

and

$$\frac{y^+}{f_i^{(2)}}\frac{\partial f_i^{(2)}}{\partial y^+}\Bigg|_\epsilon = \mu \tag{107}$$

Unfortunately, μ is a function of both ε and y, so integration is not possible. It is possible, however, to expand μ about a fixed value of the intermediate variable $\tilde{y}$. If this point is chosen properly so that $\mu = 0$, then the leading order solution is exact at this point. As distance from this point is increased, the second order term will introduce an increasing y-dependence into the profile away from the power law. The challenge to the experimentalist, of course, is to find the point of tangency, particularly since γ is not known, *a priori*. This analyis makes clear that it will be at neither fixed y^+ nor fixed $\overline{y}$, but will vary with Reynolds number.

Thus the velocity profile in the matched layer (to leading order) is a power law with coefficients and exponent which depend on Reynolds number, $\epsilon^{-1} = u_*\delta/\nu$. Since equations 102 and 103 must be asymptotically independent of Reynolds number, the coefficients and exponent must be asymptotically constant; i.e.

$$\begin{aligned} \gamma(\epsilon) &\rightarrow \gamma_\infty \\ C_o(\epsilon) &\rightarrow C_{o\infty} \\ C_i(\epsilon) &\rightarrow C_{i\infty} \end{aligned}$$

as $\epsilon \rightarrow 0$. (Note that some earlier versions of this theory included additive constants which were believed to be zero only on experimental grounds. The derivation here makes it clear that these constants are indeed zero.)

The theory, to this point at least, does not yield values or functional forms for C_o, C_i and γ. These must, therefore, be determined empirically (as with the old theory), subject to the constraint of equation 105. Of particular interest is that there are two possibilities for the asymptotic value of γ — zero or non-zero. While a non-zero value has appeal from at least an engineers viewpoint, there is also the possibility that γ simply continues to decrease as the boundary layer develops (and its Reynolds number increases) to an asymptotic value of zero. A rationale for this behavior is that the boundary layer becomes less and less inhomogeneous as it develops and approaches more and more closely the character of a homogeneous flow in x (like the channel). Seen another way, the velocity derivative in the matched layer is given by $dU/dy \sim y^{\gamma-1}$. If the limiting value of γ is zero, then the limiting value of dU/dy is y^{-1} which is the same as for a channel flow. However, the velocity profile can never be logarithmic because even an infinitesimal value of γ is sufficient to dictate a power law. It was shown by George et al. 1993 that the state of the experimental data is such that it is consistent with both zero and non-zero values asymptotic values of γ (because of the limited Reynolds number range). This question is addressed further below when the similarity of the Reynolds stress equations is considered.

It is reassuring to note if the inner and outer velocity scales are the same (as for the channel flow, v. George et al. 1993), the procedure utilized above leads to the familiar logarithmic profiles with coefficients which are Reynolds number dependent, but asymptotically constant. The possibility of Reynolds number dependent constants in the log law for pipe and channel flow is certainly contrary to the prevailing wisdom, but consistent with widespread speculation in the experimental community over many decades.

5 A New Friction Law

The relation between u_*/U_∞ and ϵ has already be established by equation 104. This friction law can be cast in more convenient form by eliminating the dependence of the right-hand side on u_*; i.e.,

$$\frac{u_*}{U_\infty} = \left(\frac{C_o}{C_i}\right)^{1/1+\gamma} \left(\frac{U_\infty \delta}{\nu}\right)^{-\gamma/(1+\gamma)} \tag{108}$$

or

$$c_f = B\left(\frac{U_\infty \delta}{\nu}\right)^{-2\gamma/(1+\gamma)} \tag{109}$$

where

$$B \equiv 2\left(\frac{C_o}{C_i}\right)^{2/(1+\gamma)} \tag{110}$$

Thus the friction law is also given by a power law, a considerably more convenient form than the implicit logarithmic relation of the old theory (equation 67).

6 The Reynolds Stress in the Matched Layer

By following the same procedure as for the velocity, the outer and inner Reynolds stress profile functions can be matched to yield

$$r_o(\overline{y};\varepsilon) = D_o(\varepsilon)\overline{y}^{\beta(\varepsilon)} \tag{111}$$

$$r_i(y^+;\varepsilon) = D_i(\varepsilon){y^+}^{\beta(\varepsilon)} \tag{112}$$

where a solution is possible only if

$$\frac{R_{so}}{R_{si}} = \frac{D_i}{D_o}\varepsilon^{-\beta} \tag{113}$$

and

$$ln(\varepsilon)\frac{d\beta}{d\varepsilon} = \frac{d}{d\varepsilon} ln\left[\frac{D_i}{D_o}\right] \tag{114}$$

Unlike the velocity, however, more information about the Reynolds stress is available for the matched layer since both equations 52 and 53 reduce to

$$\frac{\partial}{\partial y}(-< uv >) = 0 \tag{115}$$

in the limit of infinite Reynolds number. Thus, in the limit as $\varepsilon \to 0$,

$$\beta R_{so} D_o \overline{y}^{\beta} \to 0 \tag{116}$$

and

$$\beta R_{si} D_i {y^+}^{\beta} \to 0 \tag{117}$$

Since both D_o and D_i must remain finite, these conditions can be met only if

$$\beta \to 0 \tag{118}$$

Because of the presence of the $ln\varepsilon$ term in equation 114, it seems likely that β will vary quite slowly with Reynolds number, much like γ.

From equation 80 for large values of y^+, the Reynolds stress in inner variables in the matched layer is given to first order (exact in the limit) by

$$r_i = 1 \tag{119}$$

Since $R_{si} = u_*^2$, this can be consistent with equation 112 only if $D_i \to 1$ as $\varepsilon \to 0$. It follows immediately that

$$R_{so} \to \frac{D_i}{D_o} u_*^2 \tag{120}$$

where $D_o(\varepsilon) \to constant$ in the infinite Reynolds number limit.

Some insight into the behavior of $D_o(\varepsilon)$ and $D_i(\varepsilon)$ can be obtained by introducing the momentum integral equation defined by

$$\frac{d\theta}{dx} = \frac{u_*^2}{U_\infty^2} \tag{121}$$

Using this, equation 120 and the similarity relation for R_{so} from equation 86 yields

$$\frac{D_o(\varepsilon)}{D_i(\varepsilon)} = \frac{d\theta/dx}{d\delta/dx} \tag{122}$$

The relationship between θ and δ will be explored in more detail below, and it will be found that θ/δ is asymptotically constant. Thus the scale for the outer Reynolds stress is asymptotically proportional to u_*^2 as noted earlier, and the outer layer is indeed governed by two velocity scales. Note that for finite Reynolds numbers, both D_o and D_i are Reynolds number dependent so that u_*^2 alone should not be able to collapse the Reynolds stress in the matched layer, except in the limit of infinite Reynolds number. This has been observed by numerous experimenters (eg. Klewicki and Falco 1990).

7 Scaling of the Other Turbulence Quantities

For the inner layer, there is only one velocity scale, u_*, which enters the *single point* equations; therefore all *single point* statistical quantities must scale with it. This is, of course, the conventional wisdom, but with an important difference: *The inner layer ends when the velocity gradient is no longer linear!* This is contrary to the usual tendency to include the matched layer as part of the wall layer. As shown before, since the inner and outer scales are different, the matched layer variables must be expected to be functions of both, and thus Reynolds number dependent. Note that different considerations must be applied to the multi-point equations since conditions at a point can depend on those at another, and in particular those at a distance.

From the preceding analysis, it is apparent that the outer layer is governed by not one, but two velocity scales. In particular, the mean velocity and its gradients scale with U_∞, while the Reynolds stress scales with u_*^2. Therefore it is not immediately obvious how the remaining turbulence quantities should scale. In particular, do they scale with U_∞ or u_*, or both? If the latter, then quantities scaled in the traditional way with only one of them will exhibit a Reynolds number dependence and will not collapse. Note that since the ratio of velocity scales, u_*/U_∞, varies as a weak inverse power of the Reynolds number, this Reynolds number dependence would appear to reduce with increasing distance downstream and would lead to the erroneous conclusion that certain quantities scaled with only one of them take longer to reach equilibrium than others.

In view of the self-preserving nature of the outer equations for the mean flow and the preceding success with free shear flows, it is reasonable to inquire whether the equations for the other turbulence quantities also admit to fully similar solutions. The problems with treating the kinetic energy of the turbulence as a single variable have previously been noted for the plane jet. Therefore the individual components must be treated separately.

The equations for the outer part of the boundary layer are nearly the same as for the plane jet, and therefore will not be repeated here. (The boundary conditions are, of course, different.) In fact, the results are exactly the same too, except that $U_{so} \sim U_\infty$ and $R_{so} \sim U_\infty^2 d\delta/dx \sim u_*^2$ from the boundary conditions as noted above. The implications of this on the turbulence scaling has been discussed in detail by George et al. 1993 including a detailed comparison of the results with experiment. The important subject to be discussed here (and missed there) is the constraint which arises from the pressure-velocity correlations; namely that $d\delta/dx \sim constant$ for full similarity of these equations.

The most interesting of the scale constraints for the outer boundary layer (from the perspective of the earlier discussion, at least) is given by

$$\frac{d\delta}{dx} \sim \frac{D_s \delta}{U_\infty^3} \tag{123}$$

For the turbulent boundary layer $u_*/U_\infty \rightarrow 0$ as $U_\infty \delta/\nu \rightarrow \infty$ Since $d\delta/dx \sim u_*^2/U_\infty^2$, it is clear that the constancy of $d\delta/dx$ can be achieved only in the limit, and the constant is zero. This means that in the limit of infinite *local* Reynolds number the boundary layer is growing linearly with x, but with a coefficient of zero! Moreover, it is only in this limit that full similarity of the turbulence quantities can be achieved. It is easy to show that this implies that the limiting value of γ is zero, consistent with the heuristic arguments presented above.

Now what constitutes a sufficiently high Reynolds number for an experiment to reasonably approach this limit. Obviously considerable higher than ever reported, since values of u_*/U_∞ are still quite large. Moreover, at even the highest reported Reynolds numbers, the boundary layer was not growing linearly. Therefore it is clear that this limit has not been reached in experiments. It is therefore important to ask then, when should full similarity of the second order turbulence quantities be observed in the laboratory? The answer lies again in the recognition that the turbulence in the outer layer is effectively inviscid when $\varepsilon \sim q^3/\delta$. From the previous criteria, this behavior should be expected when $q^4/\varepsilon\nu > 10^4$ or approximately $u_*\delta/\nu \sim 10^4$. This is beyond the limit of existing experimental data.

It is interesting to note that the limit $\gamma \to 0$ recovers the logarithmic profile. The limiting value of the von Karman "constant" would be $\kappa = C_i\gamma$. Since $C_{i\infty}$ is finite, the limiting value of the von Karman "constant" appears to be zero. The "universal" value usually assigned to it arises from applying it to flows with Reynolds numbers which are orders of magnitude below where the profile can reasonably be assumed logarithmic. That it appears to work at all is probably due again the experimentalists efforts to make it work (usually by manipulating the shear stress and choosing the point of tangency).

8 A Composite Velocity Profile

It is possible to use the information obtained in the preceeding section to form a composite velocity profile which is valid over the entire boundary layer. This is accomplished by expressing the inner profile in outer variables, adding it to the outer profile and subtracting the common part (Van Dyke 1964). Alternatively, the outer profile could be expressed in inner variables, etc. Since the overlap region provides the common part, the composite velocity profile in outer variables is given by

$$\frac{U}{U_\infty} = [1 + f_o(\overline{y}, \varepsilon)] + \frac{u_*}{U_\infty}\left[f_i(\overline{y}/\varepsilon, \varepsilon) - C_i(\overline{y}/\varepsilon)^\gamma\right] \qquad (124)$$

Recall that f_o, f_i, C_i and γ are all functions of the Reynolds number, as is u_*/U_∞.

The composite velocity solution has the following properties:

- As $\varepsilon \to 0$ or $\delta/\eta \to \infty$, $U/U_\infty \to 1 + f_o(\overline{y})$. Thus there is a boundary layer profile even in the limit of infinite Reynolds number and it corresponds to the outer scaling law. This can be contrasted with the Millikan approach for which $U/U_\infty \to 1$, a limit remarkably like no boundary layer at all, even in its own variables.
- As $\overline{y} \to 0$, $U/U_\infty \to (u_*/U_\infty)f_i(\overline{y}/\varepsilon)$ for all δ/η. This is because the small $\overline{y}$ behavior of $[1 + F(\overline{y})]$ is cancelled out by the last term leaving only the inner solution.
- As $\overline{y}/\varepsilon \to \infty$, $U/U_\infty \to 1 + f_o$. This is because of the large $\overline{y}/\varepsilon$ behavior of f_i which is also cancelled by the last term.
- In the matched layer, only the power law profile remains.

It is an interesting exercise to substitute the composite solution into the full boundary layer equation given by equation 51. As expected, the equation reduces to equation 52 for infinite Reynolds number and to equation 53 as the wall is approached. This can be contrasted with the substitution of the Millikan law plus wake function (v. Coles 1956) in which the outer equation vanishes identically in the limit of infinite Reynolds number.

9 The Displacement and Momentum Thicknesses

The displacement thickness, δ_*, is defined by

$$U_\infty \delta_* \equiv \int_0^\infty (U_\infty - U) dy \tag{125}$$

This can be expressed using equation 124 as

$$\frac{\delta_*}{\delta} = -I_1 - I_2 R_\delta{}^{-1} \tag{126}$$

or

$$\frac{\delta}{\delta_*} = -\frac{1}{I_1}(1 + \frac{I_2}{R_{\delta_*}}) \tag{127}$$

where

$$I_1 \equiv \int_0^\infty f_o(\overline{y}) d\overline{y} \tag{128}$$

$$I_2 \equiv \int_0^\infty \left[f_i(y^+) - C_i y^{+\gamma} - B_i \right] dy^+ \tag{129}$$

and the Reynolds numbers R_δ and R_{δ_*} are defined by

$$R_\delta = \frac{U_\infty \delta}{\nu} \tag{130}$$

and

$$R_{\delta_*} = \frac{U_\infty \delta_*}{\nu} \tag{131}$$

The integrals I_1 and I_2 are functions only of the Reynolds number and become asymptotically constant; in principle, they can be evaluated from the experimental data.

The momentum thickness, θ, is defined by

$$U_\infty^2 \theta \equiv \int_0^\infty U(U_\infty - U) dy \tag{132}$$

Again using equation 124, the result is

$$\frac{\theta}{\delta} = -(I_1 + I_3) - R_\delta{}^{-1} \left[I_2 + 2I_4 + I_5 \frac{u_*}{U_\infty} \right] \tag{133}$$

or

$$\frac{\delta}{\theta} = -\frac{1}{I_1 + I_3} \left\{ 1 + R_\theta^{-1} \left[I_2 + 2I_4 + I_5 \frac{u_*}{U_\infty} \right] \right\} \tag{134}$$

where

$$R_\theta = \frac{U_\infty \theta}{\nu} \tag{135}$$

and

$$I_3 \equiv \int_0^\infty \left[f_o(\overline{y}) \right]^2 d\overline{y} \tag{136}$$

$$I_4 \equiv \int_0^\infty \left[f_i(y^+) - C_i y^{+\gamma} - B_i\right] f_o(\overline{y}) dy^+ \tag{137}$$

$$I_5 \equiv \int_0^\infty \left[f_i(y^+) - C_i y^{+\gamma} - B_i\right]^2 dy^+ \tag{138}$$

Since u_*/U_∞ varies as $(U_\infty \delta/\nu)^{-\gamma/(1+\gamma)}$ and gamma is positive but less than 1, all terms but the first vanish in the limit of infinite Reynolds number. Thus, as for the displacement thickness, the momentum thickness is also asymptotically proportional to the outer length scale, but with a different constant of proportionality.

The shape factor can be computed by taking the ratio of equations 126 and 133. The result is

$$\begin{aligned} H &\equiv \delta_*/\theta \\ &= \frac{I_1 + I_2 R_\delta^{-1}}{(I_1 + I_3) + R_\delta^{-1}(I_2 + 2I_4 + I_5 u_*/U_\infty)} \end{aligned} \tag{139}$$

For large values of Reynolds number, the asymptotic shape factor is easily seen to be given by

$$H \rightarrow \frac{I_1}{I_1 + I_3} \tag{140}$$

Note that since $f_o \leq 1$ always, it follows from their definitions that $I_1 < 0$, $I_3 > 0$ and $|I_1| > I_3$. Therefore the asymptotic shape factor is greater than unity, in contrast to the old result, but consistent with all experimental observations.

It is obvious from equations 126 and 133 that both the displacement and momentum boundary layer thicknesses are asymptotically proportional the outer length scale (or boundary layer thickness) used in the analysis. Note that it does not matter precisely how this outer length scale is determined experimentally, as long as the choice is consistent and depends on the velocity profile in the outer region of the flow (eg. $\delta_{0.99}$ versus $\delta_{0.95}$). This is quite different from the Millikan theory where the displacement thickness vanishes relative to the unspecified outer length scale. The result here is thus consistent with the experimental observation that the velocity profiles can be collapsed over most of the boundary layer by any of these choices.

10 Summary and Conclusions

The consequences of a similarity hypothesis have been explored for both turbulent free shear flows and boundary layers. In the latter case, an Asymptotic Invariance Principle was proposed which required that the properly scaled inner and outer profiles reduce to similarity solutions of the corresponding equations in the limit of infinite Reynolds number. An extension of the same principle allowed determination of the velocity profile in the matched layer and a friction law for the zero-pressure gradient turbulent boundary layer.

The turbulence Reynolds number, qL/ν, was seen to be crucial in determining whether similarity solutions were possible, especially for the turbulence quantities. Similarity solutions to the Reynolds stress equations were seen to be possible for flows where the Reynolds number was constant during the flow evolution downstream or where it increased. In both these cases, asymptotic power law type behavior is realized, although with coefficients which depend on source conditions. Flows where the Reynolds number decreased during decay were problematical in that how they behaved and for how long depended on the source conditions. The key to both increasing and decreasing local Reynolds number flows was seen to be whether the turbulence Reynolds number was sufficiently large for the dissipation to be governed by the energetic motions (i.e., $\varepsilon \sim q^3/l$) and for the energetic motions to be effectively inviscid.

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